Lectures on quantum supreme matter.

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Abstract

These notes are based on lectures serving the advanced graduate education of the Delta Institute of Theoretical Physics in the Netherlands in autumn 2021. The goal is to explain in a language that can be understood by non-specialists very recent advances in quantum information and especially string theory suggesting the existence of entirely new forms of matter. These are metallic states characterized by an extremely dense many body entanglement, requiring the supremacy of the quantum computer to be completely enumerated. The holographic duality discovered in string theory appears to be a mathematical machinery capable of computing observable properties of such matter, suggesting the presence of universal general principles governing its phenomenology. The case is developing that these principles may well apply to the highly mysterious physical properties observed in the high temperature superconductors and other strongly interacting electron systems of condensed matter physics.

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I. INTRODUCTION AND OVERVIEW.

The exceptional powers of physics in the scientific endeavour of mankind originates in the capacity of mathematics to facilitate us to look beyond the capacity of our ape brain to discern the nature of reality. The history of the subject has all along revolved around advances in mathematics and advances in quantitative experimentation that meet each other once in a while, invariably leading to scientific revolutions. In the first half of the twentieth century this was exceedingly successful, starting with Einstein's miracle year and climaxing in the landing of the standard model of high energy physics in the 1970's. But since I entered the professional floor in the early 1980's this became a sluggish affair.

But it is hanging in the air that this train may start moving again. At stake is our understanding of the general nature of matter, and given the source of the new mathematical machinery this may eventually also shed light on the nature of space-time: quantum gravity is luring around the corner. Matter is of course at the centre of the physicist's view on reality. But the way it has been understood rests on implicit assumptions that were taken as self-evident – this 'paradigm' emerged gradually and has been greatly successful in explaining the behaviour of common stuff, varying from the solids, liquids and gasses of daily life, the "quantum fluids" (metallic state, superfluids) all the way up to the Higgs field.

It is shimmering through that very different states of matter may well exist, which are not all that rare but went unrecognized for the reason that the "eyeglass" in the form of mathematical equations were not available. It is about empirical mystery meeting novel mathematics, where the latter shines a bright light on the former. The meeting place is most unexpected departing from established wisdom. Humble experimentation on messy pieces of rusted copper that started 30+ years ago (condensed matter physics) appear to communicate in a dramatic fashion with mathematical contraptions developed over a similar long period by string theorists that were originally intended to address the physics of the black hole singularity and so forth. In addition, the mathematics of information that has been evolving inspired by the quantum computer pursuit plays a crucial role as catalyzer of this reaction.

Arguably, the only serious "new math meets new physics" event that evolved during my career span has been the discovery of topological order. Quite some time ago I learned an interesting lesson from a person who played a key role in discovering the math that marries with the experiments in this context: Frank Wilczek. By shear experimental serendipity the fractional quantum Hall effects were discovered in the early 1980's. Laughlin's divine guess work surely gave away the crucial key. At the same time a mathematics propelled development was unfolding in Frank's theoretical high energy community which was rather disconnected from the tampering in the condensed matter labs. The role of the mathematical subject of topology had been picked up in the 1970's in the form of the massive Thirring model, and Frank had become aware that this had a condensed matter incarnation in the form of the Su-Schrieffer-Heeger model merely by coffee time conservations with Bob Schrieffer at the ITP Santa Barbara were both of them had a job at the time. Inspired by this, Frank wrote down topological field theory in 2+1D subsequently discovering that the mathematical topologists Chern and Simons had already accomplished this feat. The remainder is a famous history that still exerts a big influence on contemporary physics.

The advice that I got from Frank is to be acutely aware that the big leaps forward tend to happen at the border between subjects that are seemingly unrelated. He argued that there is a steady progress in these communities with facts and insights steadily accumulating but it then can happen that out of the blue there is a point where they suddenly match – CS theory being a case in point. The developments I will describe in these lecture notes may be of the same kind.

It is still work in progress – nothing is decided and it may still turn out to be a fluke. But the circumferential evidences are accumulating that we are on the right track. Moreover, it is actually a fertile ground to train young physicists for the simple reason that the separate subfields cover a wide swath of latest developments in fundamental physics.

What is this new form of matter? Only some of the most basic notions of quantum information are required to appreciate it. The main lesson of the quantum computer pursuit is that the information view on nature is clarifying. Matter is about stacking microscopic buildings together, forming a wholeness that is governed by *emergence*: the wholeness shows behaviours that are completely different from those of the parts. Macroscopic reality is constructed from quantum parts and upon staking these together one runs automatically in *exponential complexity*. The effort for a classical information processing machine increases *exponentially* in the number of parts. This is in turn rooted in the exquisite quantum-physical phenomenon of entanglement, and a machine that exploits the information processing capacity of entanglement can get it done in a polynomial time. This is captured by the quote "quantum supremacy". Matter as we know it from the textbooks of physics is computable for the reason that it is of a special kind. The macroscopic ground state ("vacuum") is actually *devoid of entanglement*, its wavefunction is called the "short ranged entangled tensor product" in the quantum info language.

The claim is that matter exists departing from a ground state that is even in the macroscopic realms characterized by the exponential complexity associated with *dense many body entanglement*. Dealing with new phenomena we need new names and I made up myself the "quantum supreme matter" quote in the title. A natural place to look for such stuff is in the context of the "non-stoquastic" (again, quantum info language) quantum systems that are naturally "infested by the fermion signs": strongly interacting electrons at finite density. This is precisely condition realized in the rusted copper materials that became famous because of their superconductivity at a high temperature, turning in a 30 year history of experimental research into a big mystery. But to even recognize it, equations are required to read the physics and given the "quantum supremacy" brick wall there used to be literary nothing lying of the kind on the shelf. But this has changed rather recently. The most important mathematical machine of the string theorists resting on $\sim$ 40 years of autonomous mathematical progress – the "AdS/CFT correspondence" – turns out to be precisely geared to compute properties of a particular form of quantum supreme matter. When one first encounters this the way it works may appear as absurd: fanciful black holes under the reign of general relativity act as quantum computers to describe the observable properties of the quantum supreme stuff. As the usual "classical" matter is governed by emergence principles such as spontaneous symmetry breaking and even the very notion of "particles" as excitations, the correspondence reveals such meta-principles governing the properties of quantum supreme matters being entirely different from nearly anything found in the textbooks – in fact, only the condensed matter main stream "stoquastic quantum criticality" comes close. In a very exciting recent development evidences are accumulating in the condensed matter laboratories that these general principles are operational in rusty copper and related systems.

My own role is very humble. It is just the dynamics of physics that brought us to the point where we are. All the great work was done in the various sub-communities: it is precisely Frank's thing. The action takes place at the borders between sub-communities. To fully appreciate one has to quite well at home in what has happened in these different communities and that is in the present stage of this affair a bottleneck. I may have myself a bit of an advantage, for the reason that my brains are of the generalist type. I am in the business of physics because of hedonism – my brains are better entertained by physics than by anything else – and as such I am also exceedingly promiscues. I am working a lot with experimentalists, but also with computational specialists. But in a way I like most to work with mathematically inclined – I stumbled somewhat by accident in the string theory business and my physics hungry brains could not get enough of it!

In my reference frame the story is easy to the degree that it appears as a no-brainer, as it should for good physics when you have finally gotten the point. I have presented it a countless number of times at general physics colloquia, leaving behind the impression that I invariably lost large part of the audiences it seems because of information overload. But most of this is just because the need to review the achievements of the various sub-tribes. For the same reason, it is intrinsically good material for an advanced graduate course since it brings together important main streams of very modern physics, acting as a case study of how research works illustrating Frank's general directives.

I decided to attempt to capture it on paper, in a maximally accessible language. Much of it is quite conceptual, and the aspects that really matter are captured by simple and elegant equations although years of intense study may be required to become a specialist in the various fields. I have to admit that this pursuit that started as an attempt to decompress my 55 minute general physics colloquium turned into a 200 page manuscript!

But much of it is devoted to the context: up to Section (V) it is all preliminaries, while the news is found mostly in Sections (VI,VII) and the rather tentative final Section (IX). String theorists can skip Section (V) and scan (VI-VIII) to look for my tweaks of the standard AdS/CFT canon. But they should carefully study Section (IV), I just know from close experience that this community has a bit of a blind spot for the sign troubles. For the computational reader Sections (III - IV) may be quite familiar. For condensed matter physicists much of Sections (II,III) may be staple food.

To help you further getting a grip on this affair, let me first present an overview of the overall narrative, followed by packing meat on these bare bones in the 170 pages or so that follow.

A. Semi-classics versus quantum supremacy.

Throughout history, studying the nature of matter has been a central subject of physics. Gradually, in the course of the twentieth century a paradigm emerged that was greatly successful in explaining a great swath of physical reality. The pillar on which this rests is best called strong emergence. One departs from very primitive microscopic building blocks, stack them together using the rules of fundamental physics and a wholeness arises showing behaviors that is entirely different from the sum of the parts. Historically, this started in the statistical physics tradition, explaining the phases of mundane matter like fluids and solids in terms of spontaneous symmetry breaking, addressing and explaining the transitions between the stable phases as well.

In the era of the standard model revolution in the 1970's it became increasingly manifest that the same kind of physical logic is also at work in the high energy realms. Symmetry is at the heart of this emergence agenda and one has to accommodate the local conservation of elementary charges employing gauge theory. This can yet be captured by the same strategy of employing primitive microscopic degrees of freedom that may even be unrelated to physical reality, eventually renormalizing in a collective physics of the right kind. The astonishing success of "lattice" QCD is case in point, in computing the difference between the proton and neutron mass departing from an in essence artificial spin system living on a lattice infused by the appropriate symmetries. There appears to be a wide spread consensus that the standard model is no more than an effective field theory, of the same kind as the "phenomenological" theories encountered in condensed matter physics such as the Ginzburg-Landau theory that impeccably describes the macroscopic behaviours of superconductors. The trouble is that the emergence is so absolute that the information regarding what is lying behind the standard model and classical gravity is completely erased from the whole. This is at the heart of the quantum gravity problem.

A priori, the complicating factor is in the fact that the microscopic building blocks are governed by *quantum physics*. A recent development that at least in my head has played a crucial role is the engineering pursuit that is presently unfolding: the construction of the quantum computer. The origin of this is in the realization that *information* as part of physical reality matters: mathematical complexity theory, devoted to a precise mathematical classification of the effort it takes to compute particular problems. This in turn revolves around the question whether this effort scales in a polynomial- ("computable") or exponential ("incomputable") fashion with the number of constituent bits. It was realized that the exquisite quantum property of *entanglement* is a computational "resource", making it possible in principle for the quantum computer that is designed to exploit the entanglement to compute exponential hard problems that are beyond the capacity of any classical computer.

This is captured by the "quantum supremacy" quote that recently made headlines.

But matter is a-priori the "stuff" formed from an infinite number of microscopic "qubits": in full generality, it *should* run into the exponential complexity associated with the *many body entanglement* in the exponentially large many body Hilbert space. But this is incomputable by conventional means. Why was it possible for our forefathers to write text books regarding the way that matter works, getting seemingly everything right? Although it still has to penetrate the high energy and condensed matter textbooks, the reasons for this formulated in the information language is actually simple and well understood. Eventually it is all in the hands of the zero temperature ground state ("vacuum") being of a special kind called "short ranged entangled tensor product states." I will review this in detail in Section (II): the vacuum is in essence "anchored" in a classical bit string. At the microscopic scale one is dealing with entanglement but in the renormalization towards the macroscopic state the entanglement fades away. The macroscopic state itself is devoid of any entanglement and can be therefore described in terms of an effective field theory in the *classical* limit.

One may then object referring to the traditional quantum liquids such as superfluid/superconductors and Fermi liquids, or even the confining state of QCD. Aren't these reflecting quantum properties on the macroscopic scale? The answer is in the microscopic *representation* used to construct the "anchor" tensor product. I will discuss this at some length in Section (II): although perhaps not generally realized, even the macroscopic Fermi liquid is to be regarded as a classical state of matter in the quantum-information sense of the word.

This macroscopic "classicalness" is actually the key to the success of the textbooks in making it possible to compute reliably the properties of much of the matter that experimentalists have encountered. This is particularly explicit in condensed matter physics where the microscopic point of departure is known: it is coincident with the mean-field theory (tied to the order parameters) which is subsequently dressed perturbatively – these "diagrams" wire in the short ranged entanglement. In high energy physics the deep UV is not known. However, it is at the core of the prevailing paradigm of quantum field theory: *semiclassics*. One departs from the *classical* field theory, in the form of an action that reproduces the classical physics at the saddlepoint. This is then requantized by promoting the free fields modes to harmonic oscillators, associating the quantized classical modes with particles. The non linearities are associated with the interactions between the particles which is then handled by the perturbative diagrams that wire in the re-emergence of entanglement limited to short distances.

This program has been very successful in charting the nature of matter in the physical universe. But does it capture everything that exists? Do states of matter exist that are incomputable with the available machinery because these are controlled by the exponential complexity inherent to the quantum physics of many things? Paraphrasing the quantum computer language, I like to call these ”quantum supreme matter”.

B. The intellectual crisis in condensed matter physics: high Tc superconductivity.

Mankind needs mathematics to deal with the physical world and without equations we are struck blind. Could it be that such quantum supreme matter is lying in front of us but we do not recognize it lacking the ”mathematical eyeglass”? The place to look for it is condensed matter physics which is after all dealing with states of matter formed from constituents that have in principle no secrets: the electron systems in solids. Different from the high energy realms, experiments are relatively easy and it is a rich basin of empirical information. It is also the realm where ”semiclassics” triumphed with the highly successful theories that are at the root of the electronics revolution (band structure) but also explaining the state of normal metals (Fermi-liquids) and superconductivity (BCS theory). In the early 1980’s the belief was widespread that the fundamentals were known.

In this constellation superconductivity at a high temperature (up to $\sim$ 150 K) was discovered in the late 1980’s in the unlikely chemical territory of oxidized copper materials. This triggered a gold rush – when this could be pushed to room temperature engineering applications would be plentiful. But it was also an era when the physicists that were young during the BCS revolution occupied the executive floor of condensed matter physics. This triggered an unprecedented hype, having the beneficial side effect that these electron system were intensely looked at in the laboratories. The prevalent mindset at the time took the existing paradigm for granted, and the focus was entirely on looking for specific variations on the established wisdom that were responsible for the superconductivity. The normal metallic state is formed from electron-like particles that in the guise of the BCS theory are subjected to a strong attractive interaction – the ”superglue” – that leads to the formation of electron pairs at a high temperature.

However, already early on experiment revealed behaviours that were greatly surprising: in an obvious way, these were at odds with anything that followed from the available equations. Halfway the 1990's the hype came to an end when it became clear that the engineering wishful dreaming would not substantiate. However, the subject stayed alive in fact driven by advances in the instrumentation in condensed matter laboratories. Although not in the public eye, this progress has been spectacular. Some of the existing "telescopes of the electron world" improved by many orders of magnitude: a case in point is photoemission where the resolution increased by a factor 10^4. Other techniques sprang into existence such as the powerful Scanning Tunneling Spectroscopy machines. These innovations were often first unleashed on the cuprates. But the more we learned, the more it became unavoidable to accept that the available equations were not delivering. This field of enquiry turned increasingly into a highly empirical pursuit. When it all started the theorists were very vocal occupying a majority of the plenary slots at the big meetings. Presently, the plenary programs are dominated by the experimentalists.

Much is going on in these electron systems. This is further complicated by the lack of an established mathematical template that can be used to filter and organize the thousands of experimental results. That is, my claim is that the recent progress in theory which is the real subject of these lecture notes actually provides to a degree such a framework. It is far from perfect, let alone that it can be claimed to be decisive but it is in a stage that I find it undeniable that the gross conceptions do shed a clear light on the mysteries. But you have to first get acquainted to this very different way of thinking about these matters. While this story unfolds underneath, I will here and there include reference to cuprate experiments, to zoom in on the big signals in experiment in the very last Section (IX).

Let me here present a short roadmap of this physics landscape, for orientational purposes [1]. It departs from the chemistry: these cuprates are formed from simple "perovskite" CuO_2 layers – you may think about them simple square lattices. These are kept apart by electronically inert space layers formed from highly ionic insulating layers. All the electronic action is in the copper oxide layers. The story start with stoechiometric "parent" compounds characterized by effectively one valence electron per CuO_2 unit cell. These are so-called Mott insulators: different from normal (band) insulators the electrons are localized because of very strong local repulsive interactions. In essence, it is just a traffic jam of electrons (Section IV D). These are subsequently doped (as in semiconductors) by dopants in the spaces layers. When the doping level p exceeds 5% or so metallic behaviour sets in and the cuprates start

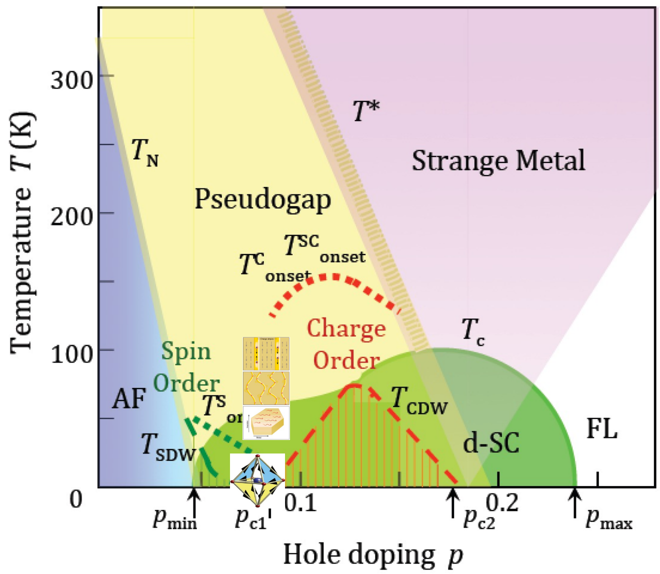

FIG. 1. The "phase diagram" of hole doped cuprate high Tc superconductivity, according to a 2015 community consensus [1]. One departs from a Mott insulator that is doped ($x - axis$) and a variety of phenomena is found as function of temperature, see the main text.

to superconduct, see Fig. (1). At doping levels less than $p_c \simeq 0.19$ one is dealing with some kind of electronic "stop-and-go" traffic and at low temperatures one finds a novel "intertwined" cocktail of exotic ordering phenomena. Ironically, I found it convenient to discuss these in a bit of a detail in the most extreme "black hole" discussion altogether, Section (VIII).

This seems to set in at the "pseudogap" temperature indicated by T^* which is decreasing in a more or less linear fashion with doping while the superconducting T_c is increasing. This reaches a maximum at the "optimally doped" doping level p_{opt}, which is close to- but not coincident with the "critical doping" $p_c \simeq 0.19$ suggested by extrapolating the T^* to zero temperature. Upon increasing doping further one enters the "overdoped" regime where T_c is decreasing. At the $p_{\max}$ superconductivity eventually disappears.

Above T^* (underdoped) or T_c (overdoped) one enters the metallic state of the cuprates: this is the king of the hill in this landscape when it comes to the big mystery story. This

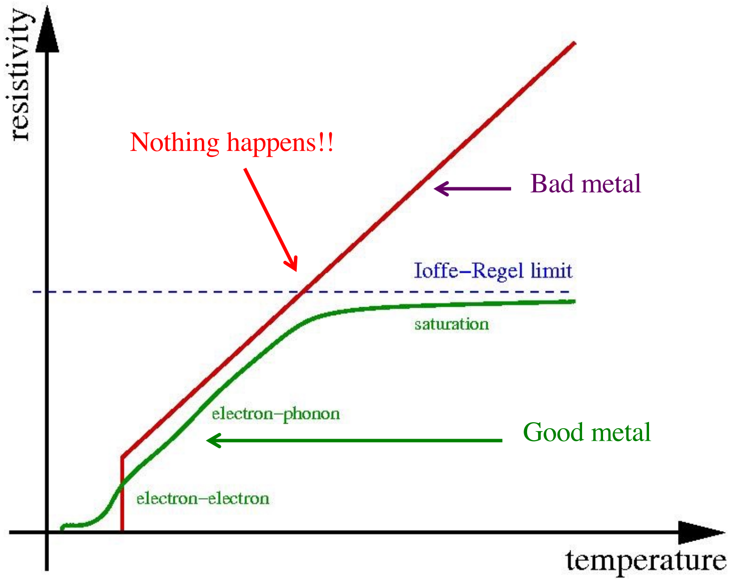

FIG. 2. The famous linear resistivity of the cuprate metal near optimal doping (red line), compared to the typical outcome for a conventional Fermi-liquid metal (green line).

was near optimal doping already established in the late 1980's and this state was called the "strange metal". This is actually the main target in the empirical theatre of the theoretical developments that I will present in these lecture notes.

The strangeness appeals to the physicist's soul (Section IX B 1). The issue is that the physical properties exhibited by the strange metals behave actually in *very simple* ways. But departing from established understanding this simplicity should not occur! This wisdom pertains to any property that has been measured but one does not have to dig deep: the DC resistivity has been all along a highlight of the strangeness, see Fig. (2). Let us first consider the resistivity of a conventional Fermi-liquid metal. At a finite temperature the Fermi-liquid cannot be distinguished from a classical kinetic gas at the macroscopic scale (Section IV B). Generically one needs a breaking of the translational invariance to obtain a finite resistivity and this is associated with the presence of the background ionic lattice. But in a Fermi-liquid the thermally quasiparticles loose their momentum individually (see Section VII). This is basically like a pin-ball machine where the scattering mechanism is by default strongly temperature dependent. At low temperature it is dominated by the "Umklapp" electron-electron scattering $\sim T^2$. Subsequently the inelastic scattering involving the phonons take over characterized by a variety of temperature dependences. Eventually the mean-free path becomes of order of the lattice constant and the resistivity saturates (Mott-Ioffe-Regel limit). But nothing of the kind happens in the strange metals: one finds a perfectly straight line all the way from the superconducting T_c up to the melting point of the crystal at a 1000 degrees or so!

This "unreasonable simplicity" reaches even farther. Optical conductivity measurements [2, 3] show that it is a so-called Drude transport, where the resistivity is set by a Drude weight ("carrier density") and a momentum relaxation rate (Section VII). The former is temperature independent while the relaxation time is $\tau_K \simeq \tau_\hbar = \hbar/(k_B T)$! This time has a fundamental status, it is just dimensional analysis: Planck's constant has the dimension of action, energy times time while $k_B T$ has the dimension of energy. I coined myself the name "Planckian dissipation" for this quantity [4]. You will meet this motive a number of times in the notes: the claim is that this is the shortest possible time associated with the production of heat, allowed by the fundamental principles of quantum physics. Moreover, it can only be reached only in a *densely many body entangled system that is scale invariant.*

It was first identified in the context of the "stoquastic" quantum critical states as discussed at length in Section (III B 4). This departs from the idea that a continuous zero temperature "quantum" phase transition occurs where some form of order disappears (see next item). Part of this agenda is that one expects a "quantum critical wedge" anchored at the zero temperature transition. At the time of writing of the Nature review [1] it was generally believed that the strange metal behaviour is driven by such a "quantum critical point" involving the disappearance of the "pseudogap order": the purple wedge enclosing the "strange metal" in Fig. (1).

But in the mean time this has changed drastically. Recent developments show that an underdoped- and overdoped *strange metal phase* exists, meeting at a first order like transition at p_c as discussed in Section (IX A). The updated phase diagram looks like Fig. (17). The central challenge addressed by these notes is to explain such "quantum critical metallic phases" in terms of a mathematical theory.

C. Stoquastic systems and the quantum critical point.

A first step in this direction that has been critical to get on the right track is the relatively well understood physics associated with "stoquastic" quantum phase transitions that I will review in Section (III). The quote "stoquastic" is yet again coming from the quantum information side. There is a special subclass of quantum systems that maps on an equivalent *statistical physics* problem. This rests on path integrals. One maps the quantum system onto the path integral "sum of worldhistories" in space time, followed by the analytic continuation to imaginary time. Under quite special conditions the ensuing path integral may become coincident with the Boltzmann partition sum. This can be handled with the formidable powers of the stochastic mathematics exploited in the statistical physics tradition: it is equivalent to addressing the occurrence of phases and phase transitions in classical systems. This territory is quite well charted.

Next to explaining the stable phases as ESR products, this also reveals that when yet other conditions are met one may find the "strongly interacting critical state" associated with thermal continuous phase transitions. The highlight is that such states exhibit universality (Section III B 3): scale invariance emerges, and the associated power law responses are characterized by "anomalous scaling dimensions". Universality means that these only depend on symmetry and dimensionality.

Although less emphasized in the textbooks, it is actually the case that the computation of the critical state runs into the exponential complexity. In the canonical quantum incarnation this means that right at the quantum critical point one is dealing with a form of *quantum supreme matter*! Paradoxically, here the "unreasonable simplicity" that begs for an explanation in e.g. the cuprates is already shimmering through: physical responses are characterized by the quantum incarnation of power laws, the "branch cuts". Simple scaling arguments reveal the generic occurrence of Planckian dissipation to be understood involving an Eigenstate Thermalization logic [5] ruled by dense many body entanglement. This is the legitimate motive for the long standing belief that the strange metal behaviour originates in such a quantum phase transition. As I just pointed out, this is not quite true in the cuprates but there is a host of other systems where the experimental support for strange metal like behaviour associated with a quantum phase transition is very convincing.

One crucial ingredient is the absence of scale: at the quantum phase transition such a scale invariance is emerging, while from the quantum-information side one learns that the presence of scales (gaps) is detrimental for the near ground state "quantum supremacy". But the bigger deal is that the quantum supremacy claim seems to imply an ultimate complexity, an exponential large number of bits that seemingly has the potential to describe an extremely *complicated* world. However, the issue is that complexity refers to the *unitary* quantum evolution that one has to tackle when one wants to compute this stuff in full. But the quantities that can be observed by experimentalists are the "vacuum expectation values" (in the jargon of high energy physics, VEV's): the *expectation values* of operators that remain after the *collapse of the wavefunction*.

This is the same deal as is at the heart of the working of the quantum computer. The unitary evolution implemented by a sequence of one- and two qubit gates processes the exponential complexity but actually information is not processed: the information processing takes place when the state collapses yielding the "read out". The observables in the physics laboratory are equivalent to the read out information. The take home message of stoquastic quantum criticality, but also the holographic agenda is that the observables exhibit an "unreasonable simplicity".

D. Quantum field theory 2.0: what lies behind the sign problem brick wall?

Especially for the mathematically inclined it is natural to stare away from problems where equations are not available. A case in point is the (fermion) *sign problem*. That there is big trouble with generic many body quantum problems was realized a long time ago. But it was effectively shuffled under the rug. In fact, all of the quantum field theory tradition in high energy physics is rooted in the stoquastic machinery that I just disussed, justifiably so since the standard model agenda is characterized by a condition (zero density) that eliminates the signs.

Computers keep people honest and the sign problem has been on the main stage of the computational community working on the condensed inspired problems. I will shortly review where they are after their 40 year of battles with this problem in Section (IV E). However, during a long stretch of my career it was put away by the mainstream as a "technical problem for software engineers". I started to appreciate that it is a *foundational* problem, perhaps even the most pressing of all "known-unknowns" in fundamental physics, already in the early

1990's and it has been with me since then. But in a recent era perceptions are changing. Yet again the quantum computer has exerted beneficial influences – its first target when it starts to work will in all likelihood be to break into the sign problem.

It cannot be emphasized enough that it is not only a programming problem: the sign problem is generic in physical systems and it acts as an impenetrable brick wall for our understanding of what is going on. There used to be no equations whatsoever! But at the same time such physical systems should exhibit reasonable physical behaviours. In fact, there is no doubt in my mind that the "strangeness" observed in the cuprates (and elsewhere in condensed matter) is just an expression of this ignorance. In fact, the Section (IV) devoted to the fermion signs is the last pre-requisite before I turn to the holographic agenda. My main claim is that *holography is the first and presently only controlled mathematical framework that is addressing a form of "sign full matter".* The outcome is that this suggests a collection of new emergence principles that may be very general governing this "physics behind the sign brick wall". These principles are entirely different from anything that is established. But these are of a remarkable elegance and simplicity themselves when you get used to them, as it should for really great physics. And they appear to be at work in a quite specific sense in the cuprates, the moral of the final section. I like to call this "quantum field theory 2.0." It is really a new release ("2"), but it is the first quite primitive version ("0").

What is the sign problem? I just explained that there is the special subcategory "stoquastic" quantum problems that can be handled by the powerful stochastic methods of Stat. Phys. with the outcome have a quite thorough understanding of the outcomes. However, this invariably involves special conditions: generic quantum problems are "non-stoquastic" in the quantum information language. The stochastic approach is failing because of the occurrence of "negative probabilities" which of course do not make sense.

Given that there is a long tradition to ignore the sign problem I have included an elementary tutorial (Section IV). Much of this is just meant to supply some basic intuition, in the form of elementary exercises such as the first quantized path integral for fermions. But there is not much to tell: I have been banging my head against it for the last quarter of a century not getting beyond some diagnostics of the trouble, being yet far away from any cure.

The crucial part is Section (IV C): the mathematical theorem by Troyer and Wiese [6] that boils down to the demonstration that the problem of strongly interacting fermions at a finite density is generically characterized by a quantum supreme ground state! This is not a complete surprise. The computational community has been battling the doped fermionic Mott-insulator problem with supercomputers for many years, without exception running eventually into the exponential complexity problem. The specific incarnation of the sign problem in doped Mott insulators can to a degree be diagnosed as I will discuss in Section (IV D).

Although not widely acknowledged in the condensed matter- and theory communities I realized it a long time ago: the problem of doped Mott insulators is characterized by a killer sign problem. Theory is empty handed explaining the cuprates, and one fact we know for sure is that these are doped Mott-insulators. The killer aspect of the sign problem is that one has to handle exponential complexity. Accordingly, one expects that the strangeness of the cuprate strange metals has dealings with the exponential complexity: the place to look for quantum supreme matter. To then find that the only machinery that does process the sign problem (holography) does suggest physics that is suggestively similar to the experimental observations. This is the take home message of these lectures.

E. Holographic duality: looking behind the quantum supremacy brick wall.

Not so long ago it used to be that we did not have even a single equation telling us anything regarding the "reasonable physics" of systems forced by the sign problem to be of the quantum supreme kind. This started to change in 2007 when out of the blue the string theory community a big effort erupted revolving around "high Tc" style condensed matter physics.

One should value the specific merit of this community. They are the contemporary representatives of the "Einstein method": seek the propulsion to decode the nature of reality in the form of *mathematics*. In the 40 years or so that string theory has been around, the mathematics that they produced is breathtaking. The trouble is that serious math cannot be explained in a tweet: one has to study it intensely, do exercises and so forth to wrap the mind around the meanings conveyed by the equations. The tragedy is that these profound equations refused to connect to empirical reality in the domain where it was intended for: quantum gravity, related matters. When a mini black hole would have banged during the first run of the LHC – a highlight prediction of string theory – it would have looked different.

Unfortunately, the LHC did not produce any surprise beyond the standard model lore.

I got myself into it in the era before 2007. I could not stand it that string theory talks appeared in my reference frame as hocus pocus and I decided to just learn it a bit. A Stanford string theory graduate came to Leiden with the desire to cross-over to condensed matter and he volunteered to organize an informal course. This great intellectual adventurer (Darius Sadri, from Persian descent) unfortunately deceased at a very young age. After a year of intense session that caused frequently headaches in my ageing brain I started to feel at home and soon thereafter the "Anti-de-Sitter/Condensed Matter Theory" (AdS/CMT) news broke. It was shear cincidence that I got into it and until the present day a main bottleneck for further dissemination is that it takes a major investment to appreciate how this math works.

The period 2007-2013 was an amazing roller coaster ride with astonishing leaps forward happening every half year or so. But around 2013 it became clear that about all low lying fruits were picked – the program was not at all done but further progress required the patience and resilience characteristic of empirical scientific pursuits. But patience is not the strength of the string theorists and in 2013 the mainstream turned suddenly elsewhere. This is best called "its form qubits" – yet again inspired by quantum information but with the focus on the quantum gravity side. This has produced some spin-off of potential relevance to condensed matter but this is revolving around quantum non-equilibrium [7], a subject beyond the scope of these lectures.

There is still much work to be done to bring the half-finished products to the condensed matter market place. But the trouble is that a work force is needed that is primarily inspired by the condensed matter context – the string theorists are excused, their heart lies with quantum gravity. But who has the time and energy in the condensed matter community to delve for a long time in this unfamiliar territory that is inherently a threat for funding success? The bottom line is that presently only quite a small group of physicists is working on the subject. These lecture notes are also intended be part of a recruitment effort. I can assure the reader that the so-called holography is very enjoyable once you have figured out how it works.

Explaining how it it all works is way beyond the scope of these notes. There are excellent books available[8–10] and one may sign up for specialized lecture courses organized by the string theorists. But the good news is that in the mean time matters have cleared up to a degree that a story can be told which is entirely focussed on the ramifications for the physics of matter. The mathematical machinery is not unlike the hardware in the experimental laboratories. As theorist one does not need all the gory details of the op-amps and other pieces of hardware that are often critical for the performance of the machines. But one can learn the gross workings of the machines, the kind of information they deliver and their shortcomings.

The machine at stake has a number of names: the "AdS/CFT correspondence", "gauge-gravity duality", or "holographic duality", while the application to condensed matter is called "AdS/CMT". This is an outgrowth of the intense pursuit in the 1980's and 1990's that climaxed in the "second string revolution" in 1995. Soon thereafter Maldacena discovered the correspondence as a spin-off. It is a no-nonsense mathematical device that makes possible to compute matters in a completely controlled way. The big deal is that it reveals a profound and surprising mathematical connection between gravitational physics – general relativity (GR) – and non-gravitational quantum physics. In the latter the "quantum supreme" part as I like to call it is on the main stage. The profundity is in the notion that this connection is "holographic", metaphorically referring to the usual holograms: two dimensional photographic plates that reconstruct three dimensional images when pierced by coherent light. The quantum physics in D space time dimensions maps on gravitational physics in D+1 dimensions.

In Section (V B) I will attempt to give an impression of how the original Maldacena AdS/CFT works, avoiding as much as possible mathematical intricacies. The take home message is that it addresses precisely the zero density stoquastic quantum criticality of Section (III), reconstructing the scaling phenomenology that I highlight in that section. This pertains especially also to the finite temperature regime where the bulk dual turns out to be governed by a Schwarzschild black hole. This is an amazing story with as highlight the "fluid gravity duality". The macroscopic behaviour of the fluid realized at a finite temperature should be governed by the Navier-Stokes theory of hydrodynamics – it has been demonstrated that there is a precise mapping between the near-horizon dynamical geometry of this black hole and hydrodynamics where the influence of the zero temperature strongly interacting quantum critical state enters in the form of the "minimal viscosity" which is the way that Planckian dissipation manifests itself in this context.

But this "holographic oracle" is still littered with mystery. It is only brought under mathematical control in a special limit of the boundary quantum physics, which has to be in the large N limit of a matrix field theory at large 't Hooft coupling as I will explain in Section (V B). Only in this limit the bulk theory turns into classical GR in the bulk – the correspondence is general but for finite N one has to tackle stringy *quantum* gravity in the bulk. Although the initial promise was that AdS/CFT would shine light on this, it is still largely in the dark. This large N limit has of course nothing to do with electrons in solids, but this does not appear to matter for the quantum criticality phenomenology that I announced in the preceding paragraph.

This unreasonable success of classical gravity to capture the phenomenological description of stoquastic densely entangled matter seems to be rooted in a deep mathematical relation. Dealing with the curved space-times of Riemannian geometry one meets the notion of "isometry", the sense of "symmetry" in this context. Consider for instance the two dimensional surface of a perfect ball. The (scalar) curvature is everywhere the same and the ball is therefore characterized by a maximal isometry. Mathematically there is a relation between the isometries of a curved manifold in D+1 dimensions and the symmetry in D dimensional flat spacetime, of the kind that controls the quantum theory. Among others it follows that the conformal invariance of the boundary theory (conformal field theory, CFT) is precisely encoded by a maximally symmetric *hyperbolic* ("anti-ball") geometry in the bulk, the Anti-de-Sitter space (AdS). One can view the correspondence as a generalized "symmetry" processing machine. More than anything else, this leads to the striking feature that the extra "radial" dimension of the bulk is coincident with the "scaling direction" of the renormalization group of the boundary theory: the RG flow is "geometrized" in the bulk, the "general relativity (GR) = renormalization group (RG)" notion.

F. Finite density and the "covariant" RG flows of strange metals.

Up to this point in the text, I have just been setting up the stage and in Section (VI) the real work starts. What has AdS/CFT to say about finite density systems, "infested" by fermion signs? Different from the zero-density case, we have nothing to compare with. There is no doubt that the matter described by the boundary is densely many body entangled – quantum supreme – but the sign problem makes it impossible to address it in any other way theoretically. But the holographic outcomes are also at finite density controlled by the

"GR = RG" principle that exerts its great power at zero density, now revealing a differently structured phenomenology. My outlook is to mobilize the condensed matter systems as "naturally occurring quantum computers" to check whether the generic, overarching RG principles suggested by this AdS/CMT holography are also under these circumstances at work.

If successful, it would shed a penetrating light on the high Tc enigma and demonstrate that radically new forms of quantum supreme matter do exist in the physical universe. But it also works the other way around. It would be a most valuable piece of information alluding to the highly mysterious aspects of the correspondence, that may be of help for the string theorists in their quantum-gravity quest. When you are a condensed matter experimentalist, realize that there is a potential that with your humble equipment you may dig out facts alluding to the quantum gravity mystery that may turn out to be way more consequential than anything that will be delivered by the high energy physicists and astronomers!

Although the jury is still out, and in a number of regards it is still quite confused, evidences have been accumulating in recent years that we are on the right track. In the final Section (IX) I will discuss a variety of recent experimental developments that appear to reflect some of the most salient, generic features of this holographic strange metal phenomenology.

This finite density holography took off in 2008. The holographic "dictionary" is insisting that it is encoded in the bulk in the form of a charged black-hole like object. All along the progress was propelled by the richness of GR that is unleashed under these conditions, there is just much more possible than for the zero density case. The correspondence is a merciless computational device and upon dualizing the precision bulk solutions a boundary world opened up that with the eyes half closed has striking similarities with what is observed experimentally in the high Tc style electron systems. But it is stronger than that – there is an eerie similarity with the Fermi-liquid/BCS wisdoms considering the gross features of this physics. This was the main motive, to a degree subconscious, that fuelled the enthusiasm. I was myself not an exception. Although there is some mention of quantum information, the prevalent sentiment you find in our book that we completed in late 2013 [9] is of this "ain't it cool that fancy black holes encode for stuffs that are eerily similar to what electrons do?"

The present perspective that it reveals general phenomenological principle alluding to fermion-sign induced dense entanglement gradually cleared up in the intervening period

[11]. This view may well be surprising (if not upsetting) for a string theorist that bailed out in the 2013 era.

One way to phrase the outcomes is to call the holographic strange metals "generalized Fermi-liquids", actually in the specific sense that the strongly interacting stoquastic quantum critical states are generalizations of the free Landau critical state above the upper critical dimension. As for the latter, the Fermi-liquid is characterized by *power law* physical responses indicating some form of absence of scale which is yet quite different from the scale invariance controlling the quantum critical point affair. The origin is in the *degeneracy* scale: the Fermi- energy. The sign problem enforces nodes in the wave function that will have the universal consequence of inducing a huge zero-point motion energy. In a Fermi-gas this is easy: fill up the single particle states employing the Pauli principle and one finds a Fermi-energy that is in common solids easily of order 10^5 Kelvins, while the ensuing Fermi-pressure keeps on the big stage of the universe neutron stars from collapsing into black holes. But this is universal: also the densely entangled incarnations have to deal with such a fermionic degeneracy scale.

The Fermi-energy is just accommodated automatically in the Fermi-liquid power laws. For instance, its (Sommerfeld) entropy $S \simeq T/E_F$. Holography is in this regard an eye opener because it spells out the origin of this different scaling behaviour exploiting the geometrized renormalization group in the bulk. This very recent insight that I learned from Blaise Gouteraux [12] is spelled out in the most important passage of these lecture notes: Section (VI), climaxing in Section (VI D 3). Instead of the bulk isometry being *invariant* under scale transformations as for the CFT's, it is *covariant* instead. This accommodates the degeneracy scale in a natural way and this principle enforces the properties of the Fermi-liquid in terms of a set of associated scaling dimensions of a kind that you don't find in a similar form in the Stat. Phys. books.

The Fermi-liquid takes now the role of the free critical fixed point and this is "deformed" in the densely entangled strange metal by turning these into *anomalous* dimensions. The claim is that such scaling flows can be completely classified unleashing powerful gravitational means. This shows that physical theories exist where the anomalous dimensions can become very anomalous. The highlight is the dynamical critical exponent expressing the scaling relation between space and time becoming *infinite*. There is direct experimental evidence for such "local quantum critical" scaling in the cuprates (Section IX B 3) and together with the Planckian dissipation I perceive this as the leading evidence that we are on the right track.

There is much more possible under the reign of scale-covariant scaling than under the highly constraining scale-invariant version of the stoquastic critical states. The bottom line is that one naturally reconstructs the portfolio of metal physics, that also includes the BCS-type instabilities of the Fermi-liquids. This turns out to be a greatly entertaining affair in the bulk. It includes spontaneous symmetry breaking in the boundary, requiring that the bulk black hole acquires "hair", a condition that is actually possible and in hindsight natural in a space-time which is asymptotically AdS. The ensuing superconductor is remarkably similar to the usual BCS variety, to the degree that the differences rooted in anomalous scaling dimensions are for practical reasons not measurable!

There are also warning signs. One has to be acutely aware that the IR "strong emergence" sector may still be constrained by the UV. At finite density one departs from the strongly interacting large N CFT affair that one then pulls to a finite density. A basic form of such "UV sensitivity" one encounters in the context of transport (Section VII) – the UV point of departure is governed by ultra-relativistic, zero rest mass matter while electrons have a rest mass that is a factor $\sim 10^8$ larger than the energy scales of interest. As I will explain, certain features in the finite density holographic transport are critically sensitive to this rest mass. But the real danger is in the large N limit that we know too well is quite unphysical. Our Leiden group has a substantial IP in the discovery of the "Leiden-MIT" fermions [13], an achievement that was a considerable stimulus early (2009) in the development. This implements the computation of "holographic photoemission", revealing Fermi-surfaces and a lot more. Quite a bit later this was however debunked as being an affair that is completely tied to the large N limit (Section VI E 4).

G. The frontier: holographic transport and intertwined order.

All along transport has been on the foreground: I already alluded to the fluid-gravity duality. In Section (VII) I will first remind the reader of the general principles underlying transport phenomena – every physicist should know this but one may check it out since especially the solid state textbooks tend to be infested with folklores (like "Drude conduction proves the Fermi-gas"). The take home message is that according to holography the finite temperature DC transport in the quantum supreme metals is invariably controlled by *hydrodynamical* flow behaviour. This is yet again rooted in the extremely fast thermalization in such systems – local equilibrium is reached well before the translational symmetry breaking becomes noticeable destroying the (conserved) total momentum. Given what we know experimentally, there is one ploy [11] that explains the linear resisitivity of Fig. (2) in fact in terms of the minimal viscosity (Section VII C 2). This is a quite predictive affair but yet again the type of experiments required to (dis)prove these assertions are yet again for practical reasons quite hard to realize – this is a relatively mature affair and several experimental groups are trying to make this work.

But we are now entering the *uncharted* part of the holographic portfolio. Electrons in solids are strongly influenced by the presence of an ionic lattice: the band structure affair. But this is also the case for the holographic strange metals and this translates to technical hardship in the gravitational bulk. This "Umklapp" deteriorates the isometry having the ramification that the hardship of the Einstein equations as a system of highly non-linear partial equations have to dealt with. This can be handled with state of the art numerical GR but only a couple of exploratory shots were fired, providing proof of principle that it can be done. This was actually an important motive for the string theorists to turn elsewhere – the required computational effort is just not something that is in their genes. Surely, the hardship of the computations goes hand in hand with the potential of finding yet other surprises. But we know very little, and this "computational holography" is still in its infancy. In Section (VII D) I will present the little we know, referring to very recent work.

The same theme is at work in the holographic symmetry breaking affair. I alluded to holographic superconductivity which is easy to compute. However, it was discovered that holographic strange metals are also subjected in a natural way to spontaneous breaking of space translations and rotations: crystallization(Section VIII). But this again kills Killing vectors and the computation of these holographic crystals again requires numerical GR. The outcomes are actually the most complex stationary black holes that have been identified. This has everything to do with the "hair" that becomes very structured. The ramification for the boundary is that this "Rasta black hole hair" dualizes in ordering phenomena that are eerily similar to the "intertwined order" found below T^* in the underdoped cuprates (Fig. 1).

H. How about experiment?

Of course, the 64K$ question is: are the cuprate strange metals of the quantum supreme kind? If so, do these give in to the general principles that are suggested by holography? As I already stressed, this is as of yet undecided. It is actually right now in a rapid flux: the experimental community has taken up an intense research effort which is quite fertile. I will present what I perceive as clear "holographic signals" in the data, but there is also a lot that is not so easy to explain. This section is rather tentative, my guts feeling is that in a period of a year or so it will need already a thorough revision.

II. LANDAU'S IRON GRIP: WHEN ENTANGLEMENT IS SHORT RANGED.

I assume that the reader will have succeeded in digesting the theoretical curriculum that is more or less standard in physics departments. Towards the end of it one learns the art of quantum field theory, the fundament of the standard model of high energy physics. It is presented as seemingly fundamental principle that one departs from a classical field theory, as formulated on basis of symmetry and locality. This is then re-quantized, in the canonical representation by lifting the classical field modes to quantum harmonic oscillators. These modes turn into particles identified by their quantum numbers. One then proceeds by incorporating the interactions using perturbation theory: the art of diagrammatics. Equivalently, in the path-integral representation one just inserts the Lagrangian of the classical theory in the action. The classical saddle point that minimizes the action coincides with the classical theory, and one requantizes the theory by expanding around the classical saddle.

In a parallel development that started in the 1930's, it became clear that the same basic structure is also at the heart of the *collective* properties of mundane forms of matter under ambient conditions. This involves macroscopic numbers of particles that themselves belong to the realms of chemistry, as described by the Schrödinger equation: electrons, ions, spins. These were the subject of statistical- and later condensed matter physics. The commonalities with the high energy realms were fully realized in the 1970's when visionaries like Ken Wilson, Sasha Polyakov and Gerard 't Hooft mobilized statistical physics wisdoms such as the renormalization group and weak-strong dualities to make progress with non-perturbative aspects as encountered especially in QCD. This went back and forth, also fertilizing condensed matter physics. Profiting from the mathematical sophistication of the high energy community, books were written having in one or the other way "quantum field theory in condensed matter" in the title.

All along the key principle in the condensed matter tradition has been what is nowadays called "strong emergence". The whole is so different from the parts that the parts can no longer be deduced from the properties of the whole. The fields of relevance to condensed matter physics only exist in a rigorous fashion in the thermodynamic limit where the number of parts goes to infinite. In the mean time there appears to be a community wide consensus that this strong emergence may also be governing principle in high energy physics. The standard model is now viewed as an effective, in essence phenomenological description that does not reveal the nature of the underlying parts. Given the many free parameters of the model itself, but also the dark sector mystery and especially the need to unify with gravity: there *has* to be a new reality beyond the present high energy border but its signals are shrouded by strong emergence in the experiments that can be done. The present state of string theory may offer a vivid illustration of this way of thinking.

However, in this pursuit semiclassics appears as the first law. This also used to be the case in condensed matter physics. In the 1970's this discipline was presented to students like myself as a theatre forming a mirror image of the high energy realms, revolving around "elementary excitations" that are like the particles of particle physics: phonons, magnons, Fermi-liquid quasipartices and so forth. But under pressure of empirical developments that started with the discovery of high Tc superconductivity in the late 1980's, in a development that took 30 years or so the semiclassics paradigm got increasingly into trouble. A myriad of mysteries were revealed by the experimentalists that appear to be completely detached from anything that can be explained by "particle physics".

As announced, these lecture notes are dedicated to yet a very different form of strong emergence giving rise to a collective quantum physics which is not governed by semiclassics. The recent progress is driven by the help arrived from the mathematical side – this venture is propelled by the good work of string theorists culminating into equations relating to a quite different reality that may be realized in the electron systems of condensed matter – I like the provocative nickname "unparticle physics" [14].

For those who are intrigued by "fundamental physics", is there any reason to pay attention to this mundane stuff in the condensed matter laboratories? Precedent from the history physics suggests one better does so. Once upon a time, van der Waals was studying the evaporation of liquids in gasses, thereby confirming that Boltzmann guessed it right. But the stochastic equations of Boltzmann turned in a much later era into the mathematical underpinnings of Yang-Mills gauge theory. Could it be that the dark sector or even the inflationary fields are unparticle physics? I would not dare to make any claim but it should be beneficial to realize that there is room for stuff that is radically different from a dilute gas formed from yet another kind of particle.

A. Short range entanglement.

The other mathematical advance has been the rise of quantum information. In fact, this is still in its infancy dealing with thermodynamically large systems. Speaking for myself, I have experienced this mathematical thinking revolving around *information* as enlightening. All one has to realize is that *entanglement* is a unique quantum physical computational resource. The basic notion is that quantum computers can compute under certain conditions *exponentially faster* than any classical computer: the "quantum supremacy" that made headlines recently.

Realizing that information matters one is urged to view matters from the standpoint of computational complexity theory. In first instance one just needs to appreciate the most basic notions and these are simple. All along we were somehow vaguely aware that we were cutting corners, but with the complexity theory at hand one can no longer stare away and one is forced to face the challenge head on. As I will forcefully argue in the next section, the generic problem of *a thermodynamic system of strongly interacting fermions at a finite density* is presently *not computable* neither by any classical supercomputer that can be build, nor by any form of established mathematics.

The first step is to appreciate the limitations of semiclassics in information language. The task is to address the question, what is the general nature of matter as formed from infinities of microscopic quantum degrees of freedom? This revolves around the nature of the zero temperature *ground state* ("vacuum" in the high energy language) – the finite temperature properties ascend from the ground state as you will see.

The information language reveals a single, completely general and very simple condition that a vacuum should satisfy in order for the physics to become semiclassical. When this is satisfied it prescribes the algorithm that tells us how to compute physical properties, a-priori ensuring success: the machinery of the physics textbooks. This condition is: *the vacuum has the structure of a short-ranged entangled product state*, abbreviated as the "SRE product vacuum".

For simplicity, let us depart from a macroscopically large system of N qu-bits, where $N \rightarrow \infty$, in fact just two level systems that may be interpreted as microscopic $S = 1/2$ spins. What follows is actually not depending in an essential way on the identity of these microscopic quantum degrees of freedom. Using this quantum computer language the Hilbert space is spanned by classical bit strings encoded quantum mechanically in a tensor product basis of the kind

$$|\text{config.}\rangle_k = \cdots \otimes |0\rangle_{i-2} \otimes |1\rangle_{i-1} \otimes |1\rangle_i \otimes |0\rangle_{i+1} \otimes |1\rangle_{i+2} \cdots \quad (1)$$

The dimension of this Hilbert space is set by all possible ways to distribute these two values over the N parts: this amounts to 2^N dimensions and this is the origin of the a-priori exponential complexity.

One expects that "typical" energy eigenstates are completely delocalized in this exponentially large Hilbert space, i.e. these are of the form

$$|\Psi\rangle_l = \sum_{k=1^{2^N}} a_k^l |\text{config.}\rangle_k \quad (2)$$

where every amplitude a_k^l is order $1/\sqrt{2^N}$: these are exponentially small.

This reveals the "quantum supremacy" troubles. As function of N the Hilbert space grows exponentially. For example, when I entered the scene in the mid 1980's the Cyber supercomputers of the day had a flop rate comparable to a cheap 2021 smartphone. Back then one could compute by brute force the exact ground state for a system (in fact $t - J$ model, see next section) of size $N \simeq 20$. Using the fastest 2021 supercomputer this has increased to $N \simeq 25$. The fun of the quantum computer is that the computational effort will scale in a polynomial fashion $\sim N^{\#}$ when sufficient computational qubits become available.

Hence, for large N such states are just not computable with classical means. Of course this also applies to analytical calculations that are particularly 'easy' viewed from the complexity angle. The teeny-weeny memory banks and millisecond flop rates of our human brains can even handle it. This intrinsic exponential complexity of the quantum many body problem is an insurmountable brick wall for our brains to comprehend how nature works in full generality.

Neverheless, the textbooks of physics are greatly successful in explaining large swathes of the matter as it occurs in the universe. How can this be? Apparently this stuff circumvents the exponential complexity brick wall. We have arrived at the instance where I can reveal the hidden assumption on which semiclassics is resting, regardless the context. Instead of the general wavefunction Eq. (2) matter that we understand is invariably characterized by the SRE product vacuum of the form,

$$|\Psi\rangle_0 = A_0 |\text{config.}\rangle_{CL} + \sum_k a_k^0 |\text{config.}\rangle_k \tag{3}$$

There is a particular "classical state" (tensor product) $|\text{config.}\rangle_{CL}$ that "dominates" the vacuum: the amplitude A_0 is finite even in the thermodynamic limit. For this to be the case the computation of the "dressing" $\sum_k a_k^0 |\text{config.}\rangle_k$ is of polynomial complexity. In fact, this is the familiar affair of computing the "fluctuations around the ground state" using the converging diagrammatic perturbation theory.

It revolves around how the many body entanglement is organized and all what matters is that $|\text{config.}\rangle_{CL}$ is an unentangled tensor product. Pending the system this can be composed from any microscopic representation. To see what this means, let us consider a simple and overly familiar example: every day solids. Surely, solids are also at zero temperature on the macroscopic scale governed by classical field theory, in fact the one longest in existence: the theory of elasticity. As everything else, departing from the microscopic scale it should have a wavefunction and it is of course obvious how to stitch this together. We depart from real space wavepackets departing from the quantum mechanics of single atoms. In second quantized representation $Y^\dagger(\vec{R}, \sigma)$ where R is the space coordinate and σ the width of the wavepacket. Since the particles are localized we can ignore the (anti)-symmetrization condition associated with indistinguishability (see next section). We now write

$$|\mathbf{R}, \{\sigma\}\rangle_{CL} = \Pi_{i=1}^N Y^\dagger(\vec{R}_i, \sigma_i) |\text{vac.}\rangle \tag{4}$$

Where $\mathbf{R} = (\vec{R}_1, \vec{R}_2, \cdots \vec{R}_N)$ is the configuration space specifying the positions of all atoms. When this forms a periodic lattice one recognizes immediately the crystal. But in order to keep the kinetic energy finite, these wave-packets require also a finite width.

However, turning to a substance formed from large mass atoms and strong inter-atomic interaction potentials (like nearly all common solids) this may become so small that it is beyond observation.

We know of course that this is not yet the full story. We learn from the undergraduate text book that the lattice vibrations of the truly classical crystal have to be requantized by promoting them to harmonic oscillators, the general procedure defining semi-classics. This will give rise to zero point motions of the atoms (quantum Debye-Waller factor), and when the crystal becomes more quantal one has to address as well phonon-phonon interactions due to anharmonicities, to eventually consider virtual processes involving topological excitations: dislocation-antidislocation loops. One recognizes the perturbative gymnastics of the textbooks, and this just amounts in the wave function language of Eq. (3) to the computation of the amplitudes a^0_k.

The unique feature of the SRE state is that the finiteness of A_0 acts like an anchor in Hilbert space, preventing the vacuum to delocalize in the full, exponentially large Hilbert space thereby rendering the perturbation theory to be convergent. A complimentary perspective is to view it from a renormalization group perspective but now keeping an eye on the entanglement. The nature of physics is "running" with scale. The $|\text{config.}\rangle_k$ imply that the product $|\text{config.}\rangle_{CL}$ gets entangled but upon zooming out to larger scales this will disappear: invariably one can identify an "entanglement length", L_{en}. This length signals that one has an effective "wave packet" which is dressed up by local entanglement, but at length scales larger than L_{en} these form together a perfect product state. This is completely devoid of any form of quantum entanglement and can therefore be described by a theory that processes exclusively *classical* information: the "effective" classical field theory.

This becomes later on a crucial insight. The "quantum supreme" vacuum states of form Eq. (2) are the fundament for phenomenological theories explaining the observations that leave room for the exponential complexity – we will see later how this works – with however the ramification that these cannot possibly be captured by the un-entangled semi-classical description.

Dealing with typical atomic solids L_{en} is much less than a lattice constant. However, we know about quite a number of circumstances where $A_0 << 1$ such that L_{en} becomes quite large. But the miracle is that as long as A_0 is finite the macroscopic physics is described by classical fields, albeit typically characterized by strongly altered "renormalized classical" parameters. This principle is coincident with the text book notion of adiabatic continuity. Departing from a limit where $A_0 \to 1$ where it is easy to derive the classical theory describing the collective state, one can deform the state (by truck loads of diagrams). The parameters of the classical theory describing the macroscopic system will be strongly affected but the structure of this theory will not change. This will end at a "thermodynamic singularity" – the instance where A_0 becomes zero.

B. The classical states of matter: from crystals to Fermi-liquids.

This is actually the title of lecture notes I wrote in the mid 1990's [15] for a course where I had the marching order to explain the Bardeen-Cooper-Schrieffer (BCS) theory. Back then quantum information just started and I was myself not aware of even the simple notions that I just explained. But I had surely figured out the somewhat implicit recipe everybody was using to compute matters in main stream condensed matter physics. I reconstructed the SRE vacuum as the base line, with the ramification that next to crystals and magnets I was forced to also call the conventional quantum liquids "classical states of matter". Back then it was halfway understood that one can get away with this designation dealing with superconductors but to call a Fermi-liquid "classical" was quite eccentric. Regarding the Fermi-liquid some physicists are still confused. But I got it right – in the next section I will outline the rock solid proof that the Fermi-liquid is devoid of any many-body entanglement.

This was inspired by pragmatism: departing from the SRE Ansatz it becomes easy to explain why we compute in condensed matter physics in the way it is done up to BCS theory. Step one: compute the energy expectation value of the product state $|\text{config.}\rangle_{CL}$ as function of the classical configuration space ($(\mathbf{R}, \{\sigma\})$ for the crystal) and minimize the energy. This coincides with the standard "Hartree-Fock" type mean field theory. Typically, but not always, one will find that quantities get expectation values corresponding with spontaneous symmetry breaking and one can identify the order parameter. In the crystal this describes the breaking of translations. It is then straightforward to develop the perturbation theory around this ordered state, and one continuous with the demonstration that this is converging. One then identifies conditions for these fluctuations to be (nearly) ignored for disparate reasons (like heavy atom crystals, or weak coupling BCS superconductors).

The issue is that pending the nature of the quantum problem different types of macroscopic "wave packets" are required to construct the "classical wavefunction". These choices are closely related to the notion of the "coherent state", quantum states that are as closely related as possible to their classical "descendents". The real space wave packets of the crystal are already case in point. Turning to spin systems, the point of departure is in the form of Heisenberg Hamiltonians $H = J\sum_{\langle i,j\rangle} \vec{S}_i \cdot \vec{S}_j$ where $\vec{S}$ are SU(2) operators with algebra $\left[S^\alpha, S^\beta\right] = i\epsilon_{\alpha\beta\gamma}S^\gamma$ where $\alpha,\beta,\gamma = x,y,z$ dealing with microscopic spins characterized by a total spin quantum number S. To construct the classical spin state, one maps the quantum spins onto classical spins (magnetic dipoles, being vectors) by using spin coherent states,

$$|\hat{\Omega}\rangle = e^{-i\phi S^z} e^{i\theta S^y} |S,S\rangle_z \tag{5}$$

where $|S,S\rangle_z$ is the max weight state in z-quantization, while θ,ϕ are the Euler angles defining points on the Bloch sphere. Such generalized coherent states are uniquely defined for microscopic constituents governed by any Lie group. The Ω's now take the role of $(\vec{R}_0, \sigma)$ in defining the classical configuration space and this maps the quantum Heisenberg models onto a classical spin system subjected to classical order.

This is all still common intuition but it becomes perhaps less obvious dealing with the conventional "quantum liquids": superfluids, superconductors and Fermi-liquids. Let us first zoom in on superfluids. Of course the Bose-Einstein condensate formed by non-interacting bosons is obviously of the kind $\sim \Pi_1^N b^\dagger_{k=0}|\text{vac}\rangle$. The big difference is of course that now the product is formed from microscopic states that are on the wave side of the particle-wave duality and in this sense the state is "quantum". In the collective state the bosons are all completely delocalized.

However, free bosons are completely pathological: physical bosons always interact. The next step is to switch on weak repulsive interactions as handled by the Bogoliubov theory, changing the Bose-Einstein condensate into a Bose condensate breaking the U(1) symmetry spontaneously such that the excitation spectrum is characterized by a genuine Goldstone boson (the phase mode or "second sound") with its linear dispersion relation. One encounters the Bogoliubov transformation for bosons where the phase mode is described by $b^\dagger_k = \cosh(u_k)a^\dagger_k - \sinh(u_k)a_{-k}$ where the $a^\dagger$ field operators creating the non-interacting bosons. The ground state is given by the condition $b_k|\Psi_{\text{Bog}}\rangle = 0$ for all k: it follows that $|0\rangle$ is a Boson coherent state formed from the $a^\dagger_k$ bosons of the form,

$$|\Psi_{\rm Bog}\rangle \sim e^{\alpha a^\dagger_{k=0} - \sum_k \frac{1}{\tanh(u_k)} a^\dagger_k a^\dagger_{-k}} |\text{vac}\rangle \tag{6}$$

in terms of the bare bosons.

The bosons offers a vivid example of the adiabatic continuity. One can go all the way to the interaction dominated limit by considering hard core bosons defined on a tight binding lattice: per site i one can either have no- or at most one boson. It is easy to show that this maps on a problem of $S = 1/2$ XY spins and it follows that the classical state describing this superfluid is,

$$|\Psi\rangle_{\rm Cl} = \Pi_i (\cos(\theta_i) + \sin(\theta_i) e^{i\phi_i} b^\dagger_i) |\text{vac}\rangle \tag{7}$$

where θ and ϕ are associated with the amplitude and phase respectively of the superconducting order parameter. The condensate is just formed from on-site Schrödinger cat state associated with zero and one boson. In this strong coupling case it is not at all a good idea to depart from single particle momentum space as for the Bogoliubov problem. But on macroscopic scales both cases are described by the same "mexican hat" Landau order parameter theory involving a complex scalar order parameter, although the numbers in the macroscopic theory are very different.

These cases all involve ground states that break symmetry spontaneously – a phenomenon in fact specific for *classical* field theories. But this need not to be the case. As we will discuss next, there are plenty of "quantum disordered" ground state that do not break symmetry while these are still of the SRE product kind.

Before we turn to this agenda, why is the Fermi liquid also "classical"? The Fermi-liquid is "anchored" by a state that is deceptive in its simplicity, the Fermi gas: $|\Psi\rangle_{\rm Cl} = \Pi_{k=0}^{k_F} c^\dagger_k |\text{vac}\rangle$. In section IV B I will sketch the precision argument showing that this Fermi-gas state is devoid of entanglement. But this fermionic "classicalness" has very different ramifications for phenomenology than the meat-and-potatoes "bosonic" order that we just discussed. As we will explain in this section, the Fermi-surface takes the role of order parameter but the physical consequences are quite different from the usual lore. One of these consequences is in the form of its weak coupling instabilities, with the BCS ground state as prime example:

$$|\Psi\rangle_{\rm BCS} = \Pi_k \left(u_k + v_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \right) |\text{vac}\rangle \tag{8}$$

A fermion pair is a boson, and the BCS wavefunction is of similar form as Eq. (7), but the pairs are forced to form in momentum space around the Fermi surface. The ramification is that the coherence length ξ (pair size) is typically very large compared to the lattice scale. This in turn suppressess the fluctuations ("A_0" is barely reduced from unity). But yet again, at distances large compared to ξ it is equally well described by the Landau order parameter theory.

III. STOQUASTIC MATTER AND THE QUANTUM CRITICAL STATE.

In this realm of physics, the most powerful theoretical machinery has been the stochastic mathematics underlying the success story of equilibrium statistical physics. This started with Boltzmann formulating the partition sum and culminated in a rigorous understanding of the phases of classical matter at finite temperatures. Presently, it is considered as a closed subject: the few loose ends are in the hands of mathematicians looking for mathematical proofs. In fact there is one millennium prize problem in this area: to derive analytically the confinement mass gap in Yang-Mills theory.

It cannot be stressed enough that the only truly general machinery available to deal with quantum many body physics is precisely this Boltzmannian affair. It is yet rather severely restricted: it can only be unleashed dealing with "sign-free" problems, and these are by themselves to a degree pathological. Special symmetries are required to "cancel the signs", and these symmetries are at best accidentally realized in nature. When the problem is "sign free" the equilibrium quantum problem can be mapped on an equivalent statistical physics problem that submits to the power of the stochastic methods. This subclass of problems was coined "stoquastic" in the quantum information community: a merger of "stochastic" and "quantum".

In fact, the standard notion of the "quantum critical state" originating at the quantum critical point associated with a zero temperature quantum phase transition is entirely resting on this kind of mathematical technology. In condensed matter physics this was put on the map by Chakravarty and co-workers in the late 1980's [16] and subsequently charted with great precision by Sachdev. This section is to be considered as an executive summary of his famous book [17]. The take home message will be: be extremely aware of the "stoquastic tunnel vision" but at the same time you will meet a first example of "quantum supreme matter” in the form of the *strongly interacting quantum critical state*. In more than one regard this sets a template for matters to come.

To understands how this works one has to familiarize one self with ”thermal field theory” as propelled by ”Euclidean” path integrals. This is standard fare on the theory floors but it is taught for no good reasons very late in the course programs. In case you have not learned this, catch up first. To my strong opinion any physics graduate in the 21-th century should be aware of this box of tricks if not only to become acutely aware of the limitations of the ”mathematical eye glass” in the hands of mankind. It is the mathematical pillar on which all of established field theory rests, metaphorically the ”Quantum Field Theory 1.X”. This thermal field theory went through a long development. The ”QFT 1.0” version was formulated by Kubo in the 1950’s: one of the unsung heroes of physics. Especially in the 1970’s it went through a rapid progression of updates (”1.X”) due to the groundbreaking insights of high energy theorists like Ken Wilson, Sasha Polyakov and Gerard ’t Hooft that also in QCD etcetera one should think like the statistical physicists.

The mapping of the quantum problem to an equivalent statistical physics exercise is actually a simple affair when one is used to the idea. It departs from the standard path-integral formulation leading to the expression for the zero temperature quantum partition sum

$$\begin{aligned} \mathcal{Z}_\hbar &= \sum_{\text{histories}} e^{\frac{iS_{\text{history}}}{\hbar}} \\ S_{\text{history}} &= \int dt\ \mathcal{L}_{\text{history}} \end{aligned} \tag{9}$$

where S is the action, and $\mathcal{L}$ the Lagrangian of the many body quantum system. One obtains the Euclidean incarnation by the seemingly simple ”Wick rotation”: continue real (Lorentzian time) analytically to ”imaginary time”: $t \to i\tau$. Kubo’s discovery is that by assigning imaginary time to be defined on a circle with radius $R_\hbar = \beta\hbar = \hbar/k_BT$ the Euclidean path integral computes finite temperature equilibrium physics as well. This is remarkably powerful: as you will see, finite temperature *classical* statistical physics arises as a special case of the zero temperature quantum case.

But the big deal is yet to come. Because of the i in front of $S/\hbar$ in Eq. (9) one is dealing with the oscillating sum in Lorentzian time which is hard to address dealing with a complicated many body action. But after the Wick rotation, the action picks up an i coming from the time integral over the Lagrangian, $dt = id\tau$. The effect is that $\exp(iS/\hbar) \to \exp(-S_E/\hbar)$: at first sight the path integral turns into a Boltzmann thermal partition sum that is just living in a space with one extra dimension: the "Euclidean" time τ. The Lagrangian of course also changes by the Wick rotation: time derivatives pick up the i, the standard relativistic kinetic term $-(\partial_t \Phi)^2 \to +(\partial_\tau \Phi)^2$. The kinetic- and potential energy terms in the Euclidean Lagrangian add up.

One now adds Kubo's time circle and the quantum partition sum becomes,

$$\begin{aligned} \mathcal{Z}_\hbar &= \sum_{\text{histories}} e^{-\frac{S_{E,\text{history}}}{\hbar}} \\ S_{E,\text{history}} &= \oint_{R_\hbar} d\tau \, \mathcal{L}_{E,\text{history}} \end{aligned} \tag{10}$$

This has clearly the structure of a stochastic Boltzmann partition sum where S_E and $\hbar$ take the role the potential energy and temperature, respectively, in setting the probability of a particular worldhistory. But this is deceptive: S_E is typically *not* a real quantity and thereby the "Euclidean" Boltzmann weights may become negative, or even complex. Surely, "negative probabilities" do not make sense and this is the sign problem. I will discuss this at length in the next section.

There is a subclass of problems where it is possible to find representations where the Euclidean action is real. A first requirement is *time reversal symmetry* – a complex Euclidean action cannot be avoided when e.g. a magnetic field is switched on. Similarly, *charge conjugation invariance* is sufficient condition: zero density problems are stoquastic, the secret behind the stunning success of lattice QCD. Next, the signs encode for sign changes in the ground state wavefunction and dealing with bosons Feynman already proved that ground state wavefunctions of bosons are generically positive definite minimizing the kinetic energy. This is behind the success of the quantum Monte Carlo (the Metropolis algorithm unleashed in Euclidean space-time) in the description of ^{4}He, see next section. Other circumstances are typically associated with accidental, unphysical circumstances: unfrustrated Heisenberg spins on a bipartite lattice, electrons that only interact with phonons, and so forth.

But when it works it is astonishing powerful. Linear response is integral part of this equilibrium agenda and this is crucial to link it to experiment – the "best" data is of the linear response kind because per construction it is avoiding any complications associated with the experiment itself. One just computes the (n) point statistical physics correlation functions in Euclidean signature, to subsequently Wick rotate back to Lorentzian time and one obtains the (n point, retarded) propagators that are central in the quantum theory. Let us put some beef on this affair by zooming in on some elementary examples.

A. The fruitfly: transversal field Ising model.

Ising spin systems have played a key role in the history of statistical physics, as the simplest models revealing the key principles. This is similar in the stoquastic quantum realms. The case in point is the Ising model in 2 space dimensions: this was exactly solved by Onsager in the 1940's demonstrating that phases of matter exist separated by a continuous phase transition. This integrability is actually a pathological condition but it does not matter for the big picture. This model is actually playing a key role in Sachdev's book [17], and I just follow his exposition in this regard.

This departs from the Ising problem of statistical physics,

$$\mathcal{Z} = \sum_{\text{config.}} e^{-H_{Ising}}$$
$$H_{Ising} = -\beta J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z \tag{11}$$

living, say, on a d+1-space dimensional hypercubic lattice with nearest neighbour couplings while σ_z can be taken to be the z Pauli matrices. What matters is that the DOF's on the site can take two values like ± 1. Every physicist has learned what this is about. At high temperature ($\beta J < 1$) this describes a disordered state while at $\beta_c J \simeq 1$ a phase transition occurs to an ordered state which breaks symmetry spontaneously. For e.g. ferromagnetic couplings ($J > 0$) one finds a twofold degenerate ground state, with the spins either all pointing up or down when $\beta J >> 1$.

By the path integral mapping in the reverse ("transfer matrix") this becomes in canonical formulation,

$$H_{QuIsing} = -J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z + B \sum_i \sigma_i^x \tag{12}$$

The "transversal field" Ising (quantum) problem: B is a magnetic field pointing in the x direction and since $\sigma^x = \sigma^+ + \sigma^-$ this is quantizing the classical Ising problem in d space dimensions.

1. *The integrability in 1+1D.*

To set the stage further, let us zoom on the special case in 1 space dimension. The key to the integrability is in a property of the Pauli matrices that is special to one dimensions: the operator identity called the Jordan-Wigner transformation,

$$\begin{aligned} \sigma_j^z &= 2a_j^\dagger a_j - 1 \\ \sigma_j^+ &= e^{-i(\pi \sum_{k=1}^{j-1} a_k^\dagger a_k)} a_j^\dagger \end{aligned} \tag{13}$$

where $a_j^\dagger$ creates a fermion at site j. Inserting this in Eq. (12) yields

$$H_{QuIsing} = -J \sum_j \left(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j + a_j^\dagger a_{j+1}^\dagger + a_{j+1} a_j - \frac{B}{J}(2a_j^\dagger a_j - 1) \right) \tag{14}$$

This is a simple bilinear problem of tight-binding electrons that is easy to diagonalize by a Bogoliubov transformation, thereby solving the problem exactly. But the relationship between the spins and the hidden free fermions is highly non-local: the Jordan-Wigner "strings" in Eq. (13) imply that the computation of a spin-spin correlator translates in an *infinite* point correlator in terms of the free fermions. Although equal time spin correlators are computable with quite some effort, it is extremely difficult to compute the unequal time (dynamical) spin susceptibilities for arbitrary couplings.

2. *The cohesive phases and quantum information.*

Let us first address the "stable" or "cohesive" phases, away from the phase transition. It is very easy to convince oneself that the ground states are of the SRE product kind. Let us consider first the limiting case $|J| >> B$. Depart from one the two fully polarized states, product states that are eigenstates of the Ising Hamiltonian. Act once with the spin flip operators to find out that the flipped spin causes two wrong exchange bonds costing an energy $\sim J$. Hence, there is a large mass gap and $|B/J|$ is a genuine small parameter regulating the perturbation expansion. Hence, the short range entanglement "corrections" in Eq. (3) $a_k^0 \simeq B/J$, falling off exponentially in higher order. In fact, we can increase B further and further and as long as there is a mass gap, staying away from the critical point, this perturbation theory will converge. This implies that the A will be finite any distance away from the "quantum critical point". The (short range) entanglement will fall off exponentially on a length scale that we will see can be identified with the correlation length/time associated with the phase transition happening in space-time.

Obviously, the same logic will apply in the the opposite limit $B >> |J|$, departing from the state where all spins are forced by the magnetic field to lie along the x quantization axis. But departing from the classical interpretation something odd is going on: large B corresponds with the high temperature limit and this is supposed to be a maximally random entropy dominated affair. How to understand this "orderly" nature revealed by the canonical formulation? The answer is in the notion of "weak-strong" duality that was discovered by Kramers and Wannier actually demonstrating that 2D Ising is self-dual: the high temperature phase is also described by an Ising model with an inverted coupling $\beta J \rightarrow 1/\beta J$ and the dual Ising spins are therefore *ordered in the high temperature limit*.

I am prejudiced that this duality notion is universal [18]. In full generality this works as follows. Departing from the ordered, symmetry broken low temperature state one can invariably identify the "operators" that are unique in destroying the order: the topological excitations, corresponding with domain walls (DW) for the Z_2 symmetry of the Ising model. When one such excitation "spans" the whole space time while it is delocalized it will destroy the infinite range correlations defining the long range order (LRO). At low temperatures these DW will occur in the form of small closed loops (in $d = 2$) but these grow in size when temperature is raised. Right at Tc these will proliferate and destroy the order. However, having the eyes fixed on the DW "disorder operators" one will find a strongly interacting system of DW's above T_c that will in turn condense and break symmetry. In $d = 2$ this is encoded by the dual Ising system of Kramers and Wannier.

The gross principle appears to be that order cannot be avoided. What appears seemingly as a completely disordered entropy dominated affair is actually a prejudice based on just observing the original spins. When one would observe the system with machinery that observes the disorder operators the conclusion would be the other way around. In turn it is a simplifying circumstance because it is rather easy to address the physics ruled by order.

This reflects the fact that in the quantum incarnation both the ordered and "quantum disordered" states are both governed by the tractable SRE products.

The self-duality is special to Ising in $1+1$ D. In three (space-time) dimensions it is a rule that at least for simple symmetries (Z_2, $U(1)$, even Galilean symmetry) one runs into local-global dualities: the dual of Ising in $d = 3$ turns out to be Ising Gauge theory. For XY (superfluids) one finds a neutral superfluid-charged superconductor duality structure (see next section). Even AdS-CFT appears to give to the rule, in a limited sense: global symmetry in the boundary turns into gauge symmetry in the bulk.

This duality "confusion" has an image in the canonical language. The issue is that we learn from the two qubit "Bell pair" agenda that entanglement should be *independent* of the representation. For example, consider the two-qubit state: $|\max\rangle = (|0\rangle_A \otimes |0\rangle_B + |0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)/2$. One could jump to the conclusion that this is a maximally entangled state. However, it can also be written as $|\max = |+\rangle_A \otimes |+\rangle_B$ in terms of the single bit representation $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and this state is of no use for quantum information processing. For two bits there is a perfect measure for entanglement in the form of the bipartite von Neumann ("entanglement") entropy obtained by computing the entropy of the reduced density matrix of one of the bits.

But there is no measure for infinite number of qubits – I find the formulation of such a universal measure of many body entanglement the number one challenge in quantum information. How can one be sure that a seemingly quantum supreme state turns into an ESR product upon identifying the appropriate microscopic interpretation? As an example, let us focus on the large B phase, a simple product in terms of x quantized spins $\cdots |1 \otimes |1\rangle^x_{i-1} \otimes |1\rangle^x_i \otimes |1\rangle^x_{i+1} \otimes \cdots$ (qubit notation). But let us now write this in terms of z-quantized spins $|1\rangle^x_i = (|0\rangle^z_i + |1\rangle^x_i)/\sqrt{2}$. Multiplying this out one finds $1/\sqrt{2^N}\sum_{k=1}^{2^N} |\mathrm{config, k}\rangle$, a state that appears to be maximally entangled! In the next section we will encounter a similar situation associated with superfluids.

3. Particle physics needs the SRE vacuum.

It is perhaps a provocative claim that the centre piece of established physics, the quantum particle as organizing principle, is just a symptom of an "unentangled" SRE vacuum state. The logic is perfectly universal but the transversal Ising model is a particularly easy stage to see how this works.

The way that linear response works is that in the measurement set up a local operator – ”flip a spin” (neutron scattering), ”remove one electron” (photoemission) – is infinitesimally sourced. This amounts to inserting a package of quantum numbers (energy, momentum, spin, $\cdots$) to measure the probability for this to succeed. The specialty of the SRE product vacuum is that it keeps this package of quantum numbers ”together” while this package delocalizes as a whole quantum-mechanically: we call this object a ”particle”.

But how to accomplish this in a quantum supreme state of matter where the ground state is already delocalized in the vast many body Hilbert space? By principle, when the ground state is quantum supreme so are all excited states: these have to be orthogonal to the ground state and this orthogonality pertains to the many body Hilbert space. Everything is entangled with everything and there is no room for the locality implicit to the notion of ”particle”. This is the no-brainer notion behind the statement that the presence of particles in the spectrum is a diagnostic for the SRE vacuum. Let us inspect how this works in the stoquastic examples.

The transverse field Ising models are a simple stage to see it at work (Fig. 3). Let us again depart from the large B limit with its trivial product state vacuum. The experimentalist may source the σ^z_i operator. Depart from the ground state where the spins are polarized in the x direction, $\sigma^z \sim \hat{\sigma}^+ + \hat{\sigma}^-$ where the ”hat” refers to the x-quantization. The propagator $\langle 0|\sigma^z(i,t)\sigma^z(j,0)|0\rangle$ turns into $G_{zz}(q,\omega) = \langle 0|\hat{\sigma}^-\hat{\sigma}^+|0\rangle_{q,\omega}$ in the frequency-momentum domain. The spectral function telling where the excitations are is then $A_{zz}(q,\omega) = \frac{1}{\pi}\text{Im}G_{zz}(q,\omega)$ according to the linear response lore.

What happens when J is finite, but small compared to B? At $t=0$ a spin in x quantization is flipped (action of $\hat{\sigma}^+$) inserting a ”triplet” $M_S = 1$ quantum number. The Ising term becomes in x quantization $\sigma^z_i\sigma^z_{i+1} \sim \hat{\sigma}^+_i\hat{\sigma}^+_{i+1} + \hat{\sigma}^+_i\hat{\sigma}^-_{i+1} + \hat{\sigma}^-_i\hat{\sigma}^+_{i+1} + \hat{\sigma}^-_i\hat{\sigma}^-_{i+1}$. The two terms in the middle just correspond with simple tight binding hoppings of the flipped spin. The propagator becomes thereby,

$$G_{zz}(q,\omega) = \frac{1}{\omega - \varepsilon_q} \tag{15}$$

just describing a free quantum mechanical particle with dispersion ε_q, the usual ”cosine” band with a minimum at the zone center ($q=0$). We have identified the particle in this product state limit.

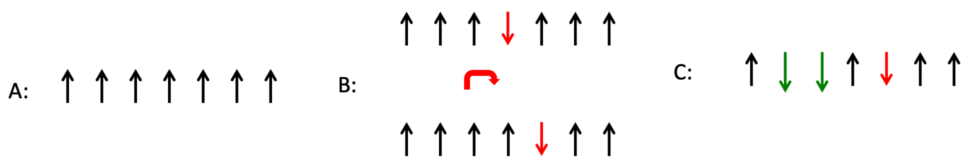

FIG. 3. The transversal field Ising model in 1+1 dimensions is a simple context to understand the emergence of quantum particles in the spectrum of a short ranged entangled vacuum state. Consider the limit $B >> J$ and the ground state approaches (A) the "classical" tensor product state: all spin are aligned by the external field. The excitation corresponds with a spin flip (B: red spin) that subsequently will hop ($\sim J$) resulting in a quantum mechanical particle characterized by a cosine dispersion relation. (C) Upon increasing J the ground state is dressed by "vacuum polarization diagrams" (green spins) but eventually the spectrum will be characterized by an infinitely long lived "particle" at the bottom of the spectrum until the quantum critical point is reached where it will turn into a branchcut "unparticle" spectrum like in Fig. 4.

But now we gradually increase J, what happens? From the form of the Hamiltonian it follows that the vacuum get dressed by increasing amounts of spin flips and the A_0 of Eq. (3) will decrease. We can keep track of it using diagrammatics and this becomes rapidly a tedious exercise. But we know the form of the outcome. As long as the perturbation theory is converging which is the case away from the critical point $G_{zz}(q,\omega) = 1/(\omega - \varepsilon_q - \Sigma(q,\omega))$ where Σ is the self-energy. Invariably one will find that close to the zone center this turns into,

$$G_{zz}(q,\omega) = \frac{A_0^2}{\omega - \hat{\varepsilon}_q} + G_{incoh}(q,\omega) \tag{16}$$

I will specify G_{incoh} underneath: what matters here is that this is characterized by a mass gap while close to the zone center a bound state will form below this cut-off and this disperses like the large B particle albeit the dispersion is renormalized to $\hat{\varepsilon}_q$ – these renormalization (”mass enhancement”) effects are in these kind of problems usually rather small. The big deal is that this quasiparticle emerges at the the very bottom of the spectrum. There are no states available for decay and this object is infinitely long lived – a perfect particle.

But the spectral weight of this quasiparticle in the spectrum is set by the product state amplitude A_0 in the SRE vacuum, becoming small upon approaching the phase transition. This is the ”quasiparticle pole strength” or ”wavefunction renormalization”' factor and this is ubiquitous. It works in the same way dealing with the quasiparticles of for instance the Fermi-liquid. This is a more complicated affair since there is no mass scale. But eventually the same infinitely long lived quasiparticle with its reduced pole strength and modified dispersion is realized in the zero energy limit.

In this particular example global symmetry is not playing a crucial role. This is different dealing with globally conserved currents. One meets here in addition principles associated with the ”wholeness” per se. A first example are the hydrodynamically ”soft” degrees of freedom rooted in *global* conservation laws, like the sound modes protected by *total* number- and *total* momentum conservation in classical fluids. But these have their quantum counterparts, like the zero-sound of Fermi-liquids but also the protected zero temperature sound modes in the quantum supreme holographic liquids. Similarly, upon breaking symmetry spontaneously Goldstone bosons have to exist. Yet again, rooted in global symmetry principle, as far as we know these will all survive in the quantum supreme systems.

Finally, let me allude to a phenomenon that is rather special for 1 space dimensions: the ”fractionalization” phenomenon. On purpose I focussed in the above on the large B phase. The same logic should apply to the large J phase as well, and in fact it does. But there is a complication. The large J phase breaks symmetry and this implies that besides the simple spin-flips there is yet another type of particle that can be identified departing from the classical limit: the ”domain wall” corresponding with a point like kink. Under the influence of B such kinks will also propagate quantum mechanically. It is easy to check that upon inserting a z spin flip this will immediately fall apart in a kink-anti kink pair both propagating independently. The spin flip is actually in this vacuum a composite particle ”fractionalizing” in two topological ”particles” that both carry half of the spin triplet quantum number: the ”spinons”. The spin-flip spectrum will be devoid of quasiparticle poles but this is just reflecting that the true particles cannot be sourced directly.

This fractionalizing phenomenon is a manifest part of the semi-classical portfolio. This simple story is to quite a degree representative for physics in one space dimension in general. At first sight this may appear as confusing: lowering dimensionality is expected to have the effect that fluctuations increase such that quantum supremacy is becoming more natural. But this is not at all the case. Generic 1+1D physics was charted already in the 1970's in the form of the Luttinger- and the Luther-Emery liquids as well as a host of spin-like systems. At least for the former two, the key is that in 1+1D interactions are *always* relevant and everything turns into algebraic long range order (ALRO) involving spin, charge density, superconducting order all at the same time. The bosons of bosonization are just the Goldstone bosons of this ALRO; a crucial aspect is surely that in 1+1D one can always transform away the sign problem, thereby avoiding quantum supremacy (next section).

The take home message of the above is that stable phases of stoquastic matter are invariably characterized by an energy scale that keeps the ground state to be a short ranged entangled product state being of polynomial complexity so that in principle it can be charted completely by classical computers. The discrete (Z_2) symmetry of the Ising systems is in this regard a simplifying circumstance. Systems characterized by a continuous symmetry may carry massless excitations but these are invariably Goldstone bosons, implied by the spontaneous symmetry breaking. But these are a trait of classical field theory that can be safely re-quantized in a semiclassical guise.

To relate this to the familiar ”mexican hat” picture, consider the Euclidian action of a complex scalar $\Psi(\vec{r}) = |\Psi(\vec{r})| e^{i\phi(\vec{r})}$ to be identified with the order parameter of a microscopic system controlled by global $U(1)$ invariance (quantum XY spins, superfluids, relativistic superconductor/Abelian Higgs field when gauged),

$$S = \int d^d x d\tau \left(|\partial_\mu \Psi|^2 + m^2 \Psi^2 + w \Psi^4 + \cdots \right) \tag{17}$$

When m^2 turns negative the *amplitude* $|\Psi|^2$ becomes finite (the mexican hat) and this is the scale responsible for localizing the ground state in the many body Hilbert space. Similarly, above the critical coupling the duality principle insists that a disorder field theory can be identified having the same effect. But the phase is still undetermined, described by an effective action $\sim (\partial_\mu \phi)^2$, the "rim" direction of the hat. Generically the phase will freeze, but only rigorously so in the thermodynamic limit: the spontaneous symmetry breaking that is well understood.

A final caveat is related to the analytic continuation of properties in Euclidean signature back to real time. This is not an issue dealing with the static (thermodynamical) properties but there is yet another difficulty dealing with *dynamical* linear responses. We learned that Euclidean signature is a greatly simplifying property, turning the stoquastic path integral in a stochastic affair. But we have to pay the prize when we want to deduce the real time dependences characterizing our quantum system. The essence is simple. Take a single, isolated excitation at energy ϵ. The two point propagator revealing its existence will be an oscillating function in the (real) time domain $\sim e^{i\epsilon t}$. But upon Wick rotation this rapidly varying function turns into a smooth exponential function $\sim e^{-\epsilon \tau}$, the object that can be computed.

This information can be retrieved dealing with a particle spectrum: this is the trick used by the lattice QCD community to find out about hadron masses, etc. However, dealing with any excitation spectrum that is more interesting than this one runs into the information loss problem. Euclidean correlators are smooth functions and tiny glitches may turn into sharp features in Lorentzian time. There are pragmatic patches to deal with this, in the first place the "maximum entropy" algorithm. But it turns out that this is a lethal circumstance dealing e.g. with the finite temperature properties at long times. Any noise in the Euclidean computation which is impossible to avoid in the absence of closed analytical solutions will amplify in an exponential fashion upon the continuation to real time.

B. The quantum critical state.

We learned in the above that the polynomial complexity is protected by the stability (rigidity) characterizing a particular phase of stoquastic matter. The only singular instance that this can fail is at the transition from one phase to the other. For this to happen the transition has to be *continuous*. In a first order transition one discontinuously jumps from one stable phase to the other.

The understanding of the "critical" state realized at such continuous transitions was of course the triumph of the renormalization group (RG) theory in the 1970's. I presume that this RG language is familiar to the reader. In fact, its powers rests on a principle that is not at all that obvious a-priori. Breaking symmetry by strong emergence is easy but *making* symmetry is a different affair. The secret of the critical state is that an immensely powerful symmetry *emerges* at the critical point: scale invariance, typically further enhanced at stoquastic phase transitions to *conformal invariance*: "angles stay the same under scale transformation", implied by the effective Lorentz invariance realized at the quantum critical point.

Although familiar from experimental observations involving systems as simple as water-steam mixtures, its profundity as emergence phenomenon cannot be stressed enough. Take the transversal Ising model: microscopically it is very scale full (the "B" and "J"). Onsager's exact solution of the 1+1D version played a crucial role in convincing man kind that it does happen: at the critical point the Jordan-Wigner fermions turn into massless Majorana's, which is easy to see by solving the simple problem Eq.(14). Absence of mass means scale invariance and this inherited also by the responses of the spins.

1. Anomalous dimensions and the branchcut response functions.

Yet again, close to the phase transition we can use the coarse grained Landau-type action, like Eq.(17) – in the case of Ising Ψ is a real scalar. The simplest way to recognize the emergent scale invariance is in terms of "Landau mean field". Just ignore the self interaction term $\propto w$ in this action and what is left behind is just a massless, Lorentz invariant free field theory $\sim (\partial_\mu \Psi)^2 + m^2 \Psi^2$, $m^2 \to 0$ which is surely conformally invariant. But free field theory is not entangled and accordingly one finds particle poles in the excitation spectrum,

$$\langle \Psi \Psi \rangle_{\vec{q},\omega} \sim \frac{1}{c^2 q^2 - \omega^2} \tag{18}$$

where c is the "velocity of light" emerging at such a quantum critical point.

But there is no a-priori reason that the self interaction can be ignored. Here the RG technology interferes. Especially in the statistical physics tradition, the role of *dimensionality* was highlighted. Given a symmetry one can identify an upper critical (space-time) dimension, D_{uc}. When $D > D_{uc}$ it is easy to establish by power counting that w is *irrelevant* towards the IR fixed point, meaning that it has completely vanished at infinite (Euclidean) times and distance. One can depart from the free fixed point and "climb up" the energy ladder by tracking the "non-universal" effects of the growing w perturbatively. This is yet again a SRE affair.

However, below D_{uc} the "self-interaction is finite at the IR fixed point" – so far it is only a diagnosis of the trouble. Although I am not aware of a rigorous mathematical proof, such a "strongly interacting critical state" appears to be generically characterized by *exponential complexity*. It is impossible to enumerate it exactly: in the quantum incarnation it corresponds with a "quantum supreme" state of matter. As such it has played a key role in this young field since the powers of conformal invariance themselves suffice to define a well organized phenomenological interpretational framework where the unknowns are just a collection of *numbers*.

The key is scale/conformal invariance. This imposes that all correlations have to be of the powerlaw kind. This is the familiar wisdom that only simple algebraic powerlaw functions are invariant under scale transformation. Change scale from x to $y = bx$ and

$$f(x) = x^{\nu}, \quad y = bx, \quad f(y) = b^{-\nu} y^{\nu} \tag{19}$$

Let us turn to the Euclidean correlators in a system that submits to conformal invariance. To keep track of time, define a distance $r_E = \sqrt{\omega_n^2 + c^2 q^2}$ in the Euclidean (Matsubara "ω_n") frequency-momentum space. A 2-point correlation function has to be of form,

$$\langle \Psi\Psi \rangle_{\vec{q},\omega_n} \sim \frac{1}{r_E^{-2\Delta_\Psi}} \tag{20}$$

using the "string theory" (conformal field theory, CFT) convention, involving the "anomalous" scaling dimension Δ_Ψ of field Ψ. Upon Wick rotation to Lorentzian time $\omega_n^2 \rightarrow -\omega^2$ and it follows,

$$\langle \Psi\Psi \rangle_{\vec{q},\omega} \sim \frac{1}{(c^2 q^2 - \omega^2)^{-\Delta_\Psi}} \tag{21}$$

This is referred to as "kinematics alone fixes the form of the two-point functions" in the CFT literature. The outcome is very simple but iconic: response functions of this form we refer to as "branch-cuts".

Obviously, when $\Delta_\Psi = -1$ we recover the free critical modes,Eq. (18) (the "unitarity limit" in the CFT jargon). However, the crucial discovery from the hay days of the criti-

$$\langle O(\vec{q},\omega)O(0)\rangle = \frac{1}{(q^2 - \omega^2/c^2)^{D-\Delta}}$$

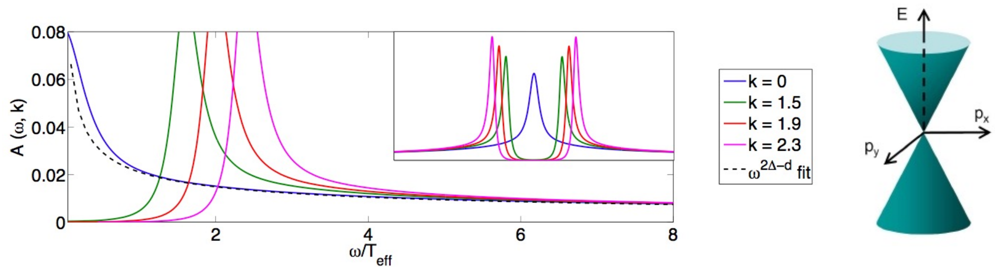

FIG. 4. A typical example of the spectral function of a strongly interacting stoquastic, conformally invariant critical state. This is the one associated with the two point fermion propagator actually computed using AdS/CFT at a small but finite temperature [13] , as would be measured by photoemission dealing with electrons. At zero momentum ($k = 0$) this is just a powerlaw as function of frequency. At finite momentum one first to enter the light-cone (right) before the spectral weight becomes finite.

cal state is that in the strongly interacting critical state realized below the upper critical dimension the scaling dimension turns "anomalous": Δ_Ψ becomes an irrational number in general bounded from the below by -1.

How does the spectrum of such a system looks like? This is illustrated In Fig. (4) dealing with Dirac–like fermions. This is actually computed using the "plain vanilla" Maldacena AdS/CFT associated with zero density. As we will see later this is just tailored to describe the *structure* of the strongly interacting quantum critical state, where one can in fact continuously tune anomalous dimensions by the free parameters of the "bottom up" phenomenological incarnation of holography.

How does this work? Let us first consider zero momentum. The propagator becomes,

$$\langle \Psi\Psi \rangle_{\vec{q}=0,\omega} \sim \frac{1}{(i\omega)^{-2\Delta_\Psi}} = \frac{1}{|\omega|^{-2\Delta_\Psi}} \left(\cos\phi_\Psi + i \sin\phi_\Psi\right) \tag{22}$$

with the phase angle $\phi = \pi\Delta_\Psi/2$. The imaginary part is the spectral function corresponding with a simple power law $1/|\omega|^{-2\Delta_\Psi}$. But the analytical structure is very tight: it follows that also the real part of the response shows this powerlaw while the relative weights of the real- and imaginary parts are also fixed via the anomalous dimension. When both can be measured the phase angle should be independent of frequency. In fact, the most convincing experimental evidence for such "strongly interacting emerging scale invariance" (there is a lot more going on than this stoquastic affair) in the cuprate strange metal is in the form of such a branch cut behaviour in their optical conductivities as I will discuss in Section (IX B 2).

One can now boost this to finite momentum. One observes that the propagator for frequencies smaller than cq: the spectral weight is vanishing outside the lightcone in the frequency-momentum domain (see Fig. 4). At $\omega = cq$ the imaginary part jumps up to then again decrease in the power law fashion.

A crucial observation is that from the analytical structure of the propagator one can directly discern whether the problem is perturbative in a free limit (the "particle physics") or whether one is dealing with a quantum supreme state: the branchcut is a first example of a signature of "unparticle physics". The belief is widespread that the self-energy form of a propagator $G(\omega, k) \sim (\omega - \varepsilon_k - \Sigma(\omega, k))^{-1}$ is universal but this is actually particle-physics folklore. Surely, the self-energy Σ may be very fanciful and difficult to compute when one has to keep track of high-order "diagrams" (in practice, vertex corrections). This is however not what matters. The issue is that the analytical structure of the self energy expression implies that it should be possible to reconstruct that one departs from a free Hamiltonian (the ε_k factor) that is remembered in the full response given the convergent perturbation theory/SRE ground state. It is obvious by comparing the branchcut expression with the self energy form that the information regarding a free limit has completely disappeared from the former.

This amounts to a warning for experimental spectroscopists. With the eyes half closed, one sees in Fig. (4) peaks dispersing in a way that looks like free Dirac fermions. It is more or less a community habit to then jump to the conclusion that this reveals band-structure electrons in the case of the most abundant form of spectroscopy: photoemission. But the information telling us whether it is particle- or unparticle physics is actually encoded in the analytical form of the propagators, i.e. the lineshapes. In fact, when one has access to both the real and imaginary part of the propagators it is quite easy to construct a "quasiparticle detector" in the form of a rigorous algorithm that will tell whether any information regarding a free limit is hidden in the experimental outcome. Although the experimental information falls short for a definitive proof, line shapes extracted from very recent high resolution ARPES data indicate that the most "particle physics" like features seen in cuprate strange metals (the "nodal fermions") are actually not passing the quasiparticle detection machinery, see Section (IX C).

As a final remark, it is a habit to call responses characterized by quasiparticles "coherent" while the "unparticle" spectra are called *incoherent"*. This is actually a rather awkward semantics. Coherence refers to wave phenomena, like a laser beam is a form of coherent light. This is actually rooted in the quantum mechanical coherence of *single, non interacting particles*. Branchcuts etcetera are also rooted in quantum physical coherence but now present in the many body Hilbert space. This terminology gets really painful in the context of transport. I already alluded to hydrodynamically modes protected by global symmetry like sound waves. Such "particles" have nothing to do with quantum-mechanical coherence, they are just rooted in global conservation laws. As I will explain later, these are behind the completely ubiquitous Drude transport. There is a habit to call these also "coherent" which is just very irritating, reflecting the effect of 100 years living in the particle physics tunnel. For this reason, I will be obnoxious: I will call "coherent" and "incoherent" instead "particle" and "unparticle" when this is not governed by global conservation law. When global symmetry is at work I will name the modes referring specifically to the conservation laws, like "zero sound" giving rise to Drude phenomenology.

2. The strongly interacting critical point and exponential complexity.

Let us return to the fundamentals: the phenomenological signature of the strongly interacting quantum critical state is in the form of the anomalous scaling dimensions. What has this to do with quantum supremacy? We can rely on the statistical physics incarnation of this critical state; although subject of intense study in an era that mathematical complexity was not on the foreground it appears that nowadays there is a community consensus that it is a NP-hard affair. I am not aware of a proof of this statement but there is an overwhelming amount of circumferential evidence for it.

The most stunning trait of the strongly interacting critical state is the property of *universality*. This encapsulates perhaps the most extreme incarnation of strong emergence leading to extreme simplicity and elegance, where the messy microscopic physics is completely washed away by the collective. Given the powers of the emerging conformal invariance, the only data specifying the systems are the anomalous dimensions – and as it turns out the three point "operator product expansion coefficients" (OPE's) are also required for high point functions. Universality means that these are determined merely by *symmetry and dimensionality*. This is the famous affair that the critical exponents are the same for the critical points of Ising spins and the water-steam system.

What is the origin of universality? Why is this "ultimate emergence" only realized at strongly interacting critical points? The literature is in this regard not overly explicit. The way it is reasoned is that it is observed in various way to then proceed arguing that it is captured by RG. But the notion of renormalization is very generic and the critical state RG is constructed resting implicitly on the presence of universality. I dare to conjecture that the exponential complexity of this critical state is actually a *necessary* condition for universality to set in. The NP-hardness erases all information regarding the short distance degrees of freedom, while in combination with the conformal invariance physical observables are all governed by the very simple scaling dimensions.

This notion will play a crucial role when we turn to the finite density, non-stoquastic holographic strange metals in the next sections. Given that the UV physics hardwired in the AdS/CFT correspondence is utterly different from the "quantum chemistry" of electrons in solids, a similar extreme form of strong emergence is required for the former to be relevant. But yet again one can safely assume that the holographic strange metals are characterized by the quantum supremacy of their ground state wave functions, while the physical properties are in the grip of emergent scale invariance with anomalous dimensions in the driver seat. However, due to the fermion signs the scaling theory revealed by the holographic "geometrization of the renormalization group" has a completely different structure than the stoquastic critical state. My "central conjecture"is therefore that in this context a very similar notion of universality is at work as in the conventional setting of continuous phase transitions, although the scaling theory itself is organized in a very different manner:

see Section (VI).

3. The singular thermal critical point and Kadanoff's scaling theory.

Let me remind the reader of the way that the thermal critical state is organized and how this translates to the physics of the stoquastic quantum critical state – this is among others a useful template to compare with when we are dealing with the non-stoquastic versions. This is eventually all anchored on the existence of the *critical point*: in order to realize the full exponential complexity an *infinite fine tuning* is required. The primary control parameter correspond with the reduced temperature $\delta t = |T - T_c|/T_c$ and reduced critical coupling $\delta g = |g - g_c|/g_c$ in the thermal- and quantum incarnations, respectively. To reach the critical state these have to be tuned *precisely* to zero: $\delta t, \delta g = 0$. The branchcuts discussed in Section (III B 1) are characteristic for the critical state and are only realized to the lowest frequencies after this impossible feat of infinite fine tuning. The associated anomalous dimensions are called the "correlation function exponent(s)", designated by η in the critical state literature.

In fact, all other anomalous dimensions are tied to the *approach* to the critical point. Looking at thermodynamical quantities such as the specific heat, these will typically show a power law singularity upon varying temperature through the critical temperature. Next to specific heat characterized by the dimension α ($C \sim \delta t^{-\alpha}$) one typically deals with the order parameter and susceptibility exponents β, γ as well as the source field exponent δ (at $\delta t = 0$). Finally, there is the correlation length ξ_{cor} that is diverging algebraically upon approaching the critical point $\sim 1/\delta t^{\nu}$. This is the crucial quantity dealing with the critical points. The way to understand it is that at shorter distances the system has already established the critical state. However, the fact that one is not right at δt has the effect that at length ξ_{cor} the relevant operator that drags the system to the stable high- or low temperature state takes over, and at scales larger than ξ_{cor} the stable states re-emerge.

This sets the "organization" of the critical state than can already be read off the mean-field Landau theory above D_{uc}. Historically, the "Rosetta stone" that lead to the decoding of the strongly interacting critical state was embodied by Kadanoff's scaling theory. Departing from the scaling form of the free energy he demonstrated that the thermodynamical scaling dimensions are all related through the scaling dimension y_t associated with the reduced temperature $\sim (\delta_t l)^{y_t}$ and the y_h associated with the field sourcing the order parameter $(h_l)^{y_h}$ (like the magnetic field dealing with a ferromagnet). It follows from elementary scaling considerations that $\alpha = 2 - d/y_t, \beta = (d - y_h)/y_t, \gamma = (2y_h - d)/y_t$ for the specific heat, magnetization and susceptibility exponents respectively.

Assuming in addition *hyperscaling* rooted in the scaling of the free energy with the volume of the system one finds that also the "geometrical" exponents ν and η are governed by y_t, y_h, implying relations with the thermodynamic exponents such as $\nu d = 2 - \alpha = 2\beta + \gamma$ and $2 - \eta = \gamma/\nu$. A crucial finding is that this hidden simplicity is only present in the strongly interacting critical state below the upper critical dimension. Departing from the free fixed point the "irrelevant operators" that switch on upon ascending from the fixed point will give rise to perturbative corrections that invalidate universality. Thus, dealing with the universality of the strongly interacting critical state all one needs to know are the y_t, y_h scaling dimensions that are in turn completely determined by symmetry and dimensionality.

This story continues with the construction of several methods to estimate these scaling dimensions. The ϵ expansion departs from the theory at D_{uc} computing perturbatively the corrections in $d = D_{uc} - \epsilon$. Famously, this is an asymptotic expansion that will fail eventually for any finite epsilon. Similarly, the computational community designed clever Metropolis updating schemes to get as closely as possible the critical point avoiding the critical slowing down (the exponential rise of the computational time before an update is accepted). In this way one can get quite good estimates for the anomalous scaling dimensions but it is presently widely acknowledged that the precise computation of these dimensions involves exponential complexity.

This spawned subsequently an impressive mathematical effort. Resting on the canonical formalism the algebraic structures associated with conformal invariance were identified: the Conformal Field Theories (CFT's). These reveal properties like the central charge, while it can be demonstrated that next to the scaling dimensions of the fields the data on three point operator product expansion (OPE's) suffice to compute all (multi-point) correlation functions. In addition, you already learned that the Ising model in two overall dimensions can be exactly solved by employing the simple Jordan-Wigner transformation. This "integrability" turns out to be generic for 2D CFT's, since the conformal mappings imply the presence of an infinite number of conservation laws. Although these represent an interesting play ground for mathematicians one should be acutely aware that integrability is a quite pathological condition dealing with the real higher dimensional world.

4. The stoquastic quantum critical point.

Once again, the quantum criticality associated with the quantum phase transition is defined by "dumping" the strongly interacting thermal critical state I just reviewed in Euclidean space time. The particularities of the quantum incarnation then just follow from Wick rotating back to Lorentzian time. Obviously, the exponential complexity associated with the sampling of the classical configuration space translates into a coherent superposition involving an extensive part of the many body Hilbert space: the strongly interacting quantum critical state is therefore an example of *quantum supreme matter* – in fact, the only example where we can claim a fair understanding.

Departing from the thermal classical state, there is one more scaling dimension that is really special for the quantum critical state: the *dynamical critical exponent* z expressing how space and time are scaling relative to each other. This can also be identified at thermal transitions but the meaning in the quantum case is completely different. By just asserting that the quantum critical state is obtained by inserting the thermal state in Euclidean space time implicitly assumes that the Eucidean space-time is isotropic: the imaginary time direction is just like a space dimension. This coincides with Lorentz invariance: the Euclidean rotations turn into Lorentz boost in real time. However, this does not have to be the case. In a stoquastic setting one can encounter for instance the situation that the order parameter fluctuations can decay in a heat bath of other excitations. Integrating these out yields an Euclidean action where the space- and time gradient terms are no longer appearing in the same footing, see for instance Section (III D). This will have the ramification that the scaling of time relative to spatial scale transformations acquires the form $t = l^z$. For instance, diffusion means $z = 2$. This is equivalent to asserting that there is not one but z time directions: the total dimensionality of space-time is $d + z$. We will see that z is in the lime light dealing with the non-stoquastic critical matter, where z can take seemingly absurd (from a stoquastic viewpoint) values including infinity.

What can an experimentalist expect to observe dealing with stoquastic quantum criticality? In Section (III B 1) we already high lighted that this quantum supremacy imprints on zero-temperature response functions by turning these into the "unparticle" branchcuts.

This is quite generic – just reflecting power law correlations controlled by anomalous dimensions – but in other regards these quantum phase transitions reveal a rather specific phenomenology.

This affair revolves around the special, singular "quantum critical point" (QCP). The experimentalists should be in the position to tune the zero temperature coupling constant controlling the quantum fluctuations. This may be a magnetic field, pressure or a chemical potential. Away from the critical coupling g_c he/she will be in the position to identify a (dual) order parameter. Upon approaching the QPT the order parameter rigidity (elastic moduli, superfluid stiffness etc) will decline but at sufficiently long times, low temperatures and so forth it will give in to the thermal physics described by classical theory: the SRE product moral, called "renormalized classical regime".

Dealing with quantum physics, space and time are "intertwined" and to get a clear view one needs *dynamical* response functions: the ideal observables are the dynamical linear response functions that at zero temperature and precisely at the critical coupling will reveal the branch cuts. But what to expect when δg is small but finite? The crucial quantity is the correlation length/time: at distances larger than ξ_{cor} the system is in the renormalized classical regime but the critical state is re-entered on scales smaller than the correlation length. In fact, time matters most and it is just convention to define ν to be associated with length, such that the correlation time $\tau_{\text{cor}} = 1/(\delta g)^{\nu z}$. Hence, at an energy $\hbar/\tau_{\text{cor}}$ the response crosses over from the dynamics of the classical theory (e.g., exhibiting Goldstone bosons) back to the critical branch cut affair at higher energy. In fact, the crossover functions are also supposed to be governed by universality but they are in practice hard to compute. The take home message is that in the idealized situation of zero temperature, full control over δg and spectroscopic machinery giving full access to the dynamical susceptibilities, the experimentalist could measure the *quantum critical wedge*, see Fig.'s (5,6). Upon moving away from the QCP he/she will see that the crossover line governed by $(\delta g)^{\nu z}$ is gradually moving to higher energy. The thermodynamic exponents of the thermal case translate in measurable quantum-thermodynamic properties and in this way one has as a matter of principle actually a quite convenient way of determining the precise nature of this quantum critical state in experiment. It turns out to be in practice a lot less easy, but this has to do with real life complications.

C. The finite temperature physics near the quantum phase transition.

Up to this point I addressed the physics at strictly zero temperature. Using the Euclidean path integrals it is actually quite easy to find out what happens at *finite* temperature: we just have to roll up the imaginary time direction in a circle with radius $R_\tau = \hbar/(k_B T)$ where T is the physical temperature. This in fact a geometrical operation and it is reasonable well understood what has to be done in the statistical physics incarnation: the art of finite size scaling which is integral part of the RG portfolio! All one has to do is to compute the influence of the finite time circle on the statistical physics correlators, to then analytically continue to real time to establish the physical linear response functions.

This is easier said than done because this is a first instance where the analytic continuation information loss problem surfaces. Let us depart again from the statistical physics of the strongly interacting critical state in Euclidean space time. Right at the critical point the system acquires the conformal invariance and one can now ask the question how correlation functions are influenced when one of the dimensions (i.e., Eucidean time) becomes finite, thereby lifting the scale invariance. Correlations at a distance r should become scaling functions of r/L where L is the finite size. Associating r with the Euclidean time direction, and realizing that the "duration" of imaginary becomes $L_\tau = \hbar/k_B T$, the issue is that the Wick rotation maintains locality in the sense that a certain duration of imaginary time is in one-to-one correspondence to the same duration in Lorentzian time. This implies that real frequency linear response functions should submit to a scaling behaviour,

$$\chi_{\Psi,\Psi}(\omega, T) = \frac{1}{T^{\Delta_\Psi}} F_\Psi(\frac{\hbar\omega}{k_B T}) \tag{23}$$

dealing with non-conserved quantities – 'hydrodynamically protected" quantities are more tricky and we will return to this in the next section. A next wisdom of statistical physics is that the cross-over function $F_\Psi(x)$ is also universal, set by symmetry and dimensionality. But these are *not* available in closed analytical form dealing with non-integrable critical states – AdS/CFT is the exception (see next section).

Why is this a problem? The Wick rotation back to Lorentzian signature is the culprit. Tiny "bumps" in the smooth Euclidean correlators may turn into rapid variations in the Lorentzian responses. This information-loss problem becomes particularly severe at frequencies small compared to temperature; small noise in the numerics of e.g. the QMC explode upon analytical continuation. Resting on elementary finite size size reasoning it is however still possible to arrive at a variety of generic behaviours associated with the finite temperature physics – this was first realized by Chakravarty et al. [16] in the specific context of magnetic systems close to a quantum phase transition. This was further eloborated by Subir Sachdev – it is a central theme in his book [17].

1. The E/T scaling and the quantum critical wedge.

A first very simple but phenomenologically powerful affair is the fact that dynamical response functions give in to the so-called "energy-temperature" ($\hbar\omega/k_BT$ or "ω/T") scaling. This is just implied by Eq. (23). Experimentalists can measure the response function over a large frequency range at many different temperatures. By plotting $T^{\Delta_\Psi}\chi_{\Psi,\Psi}(\omega,T)$ as function of $\hbar\omega/k_BT$ all the data will collapse on a single curve (representing F) for the correct value of Δ_Ψ. This Δ_Ψ should then in turn be consistent with the branch-cut setting in at $\hbar\omega >> k_BT$. This is very powerful: scaling collapses are the primary weaponry dealing also with the thermal critical state but in the quantum context the "temperature" finite size scaling becomes very convenient. Just dial up the heating! When one finds this ω/T scaling one can be sure that one is dealing with a quantum critical system. Yet again, in practice this is less glorious because of the difficulties to measure dynamical response functions over the required dynamical range.

Let us now turn to the regime where the measurement time is *long* as compared to R_τ, in other words $\hbar\omega << k_BT$, which includes DC measurements ($\omega = 0$). We imagine that the experimentalist can vary the coupling constant δg through the QCP while he has access to a large temperature range. He/she will encounter yet another fingerprint signalling that a quantum phase transition is at work. This "quantum critical wedge" is in everyday laboratory practice actually the most important indication for the claim that such physics is at work.

I already alluded to the role of the correlation "time" that is decreasing algebraically with δg upon moving away from the QCP. I explained that this will be visible in the zero temperature dynamical response functions. But the principles of finite size scaling also imply that it should be visible at zero frequency when temperature is varied. This is again very simple. When one is close but not at the critical point, at Euclidean times smaller than

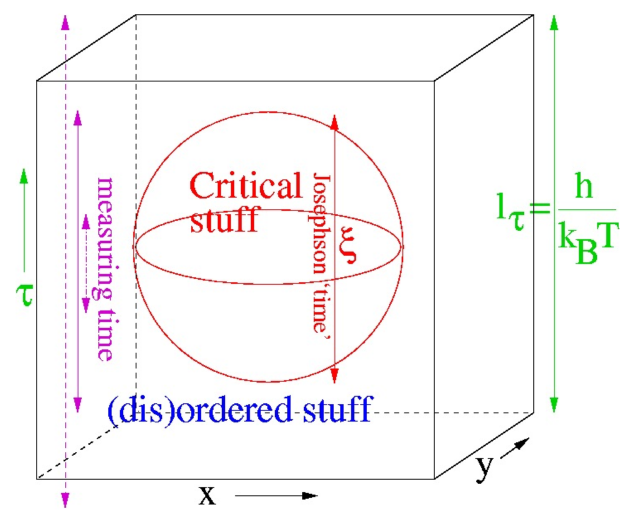

FIG. 5. Euclidean space time "road map" dealing with a stoquastic critical state. Upon approaching the quantum phase transition the "Josephson correlation time" ξ_τ increases algebraically in terms of the reduced coupling $\delta g = |g - g_c|/g_c$ to diverge at the QCP. The meaning of this is that at times shorter than ξ_τ the system behaves as if it is right at the critical point (red "ball") characterized by the Planckian dissipation, branch cuts and so forth. When the time associated with the measurement is short compared to l_τ the response will be as if temperature is zero. However, when $\delta g \neq 0$ upon raising temperature the system will be first governed by the stable (ESR product) state and instead one will find the "renormalized classical response" until $l_\tau \simeq \xi_\tau$ where it will crossover to the quantum critical "Planckian" regime. This is the explanation for the "quantum critical wedge", Fig. (6).

the correlation time $\tau_{\rm cor}$ the system behaves as if it is precisely at the critical point. Upon increasing temperature the time circle is shrinking and when its radius becomes of order of $\tau_{\rm cor}$ the finite temperature system re-enters the critical state! Hence, one expects a cross over from the long time finite temperature renormalized classical physics to the finite temperature physics one finds right at the QCP. One expects a cross over to become detectable in DC physics at a cross over temperature,

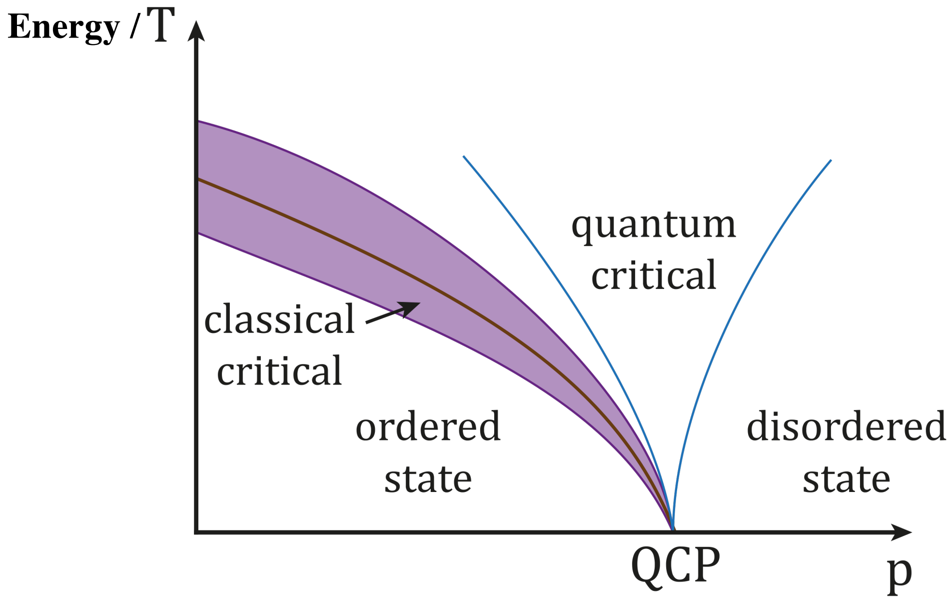

FIG. 6. The "quantum critical wedge": upon tuning the system with a zero temperature control parameter (p, like a magnetic field, pressure, thermodynamical potential, etc) one encounters a quantum phase transition at a critical coupling g_c. On both sides the physics of the stable (semi) classical states will be in charge at sufficiently low energy and/or temperature. This includes the possibility of a *thermal* phase transition (purple). However, as explained by Fig. (5) upon raising temperature the cross over to the quantum critical state will take place characterized by the Planckian dissipation. The crossover lines in the (p, T) plane are determined by the correlation length exponent.

$$k_B T_c \sim (\delta g)^{z\nu} \tag{24}$$

This defines a "wedge" in the coupling constant - temperature plane (Fig. 6), and one reads of the combination of the anomalous dimensions $z\nu$.

Although quite simple, this is empirically very powerful. In the mean-time there is a plethora of such quantum critical wedges identified in a variety of experimental incarnations (heavy fermion systems, pnictides, structural transitions) while for a long time it was taken for granted that such a quantum phase transition occurring at optimal doping would also be responsible for the "strangeness" of the cuprates. In a very recent development hard evidence appeared that this is not quite the case as I will discuss in Section (IX A).

When I discussed the Kadanoff scaling thermodynamical quantities were on the foreground. The above discussion concentrates however on the geometrical exponents ("η" ν, z). As it turns out the workings of the thermodynamics at the quantum critical point did not attract much attention, largely because it is not easy to get an experimental handle. However, this is an entertaining by itself which is worked out in full for a density driven QCP in Ref. [19].

But now the question arises, what is special about the finite temperature – low frequency physics of the quantum critical state itself? The revelation is in the realization that the classical dissipative state that ascends from the zero temperature strongly interacting critical state appears to be the *best heater* that can be realized in nature. This is the notion of "Planckian dissipation".

2. *The finite temperature quantum critical fluid: the Planckian dissipation.*

The story that follows is actually a small and simple part of a recent exciting development, indicated by "Eigenstate thermalization". This also includes non-equilibrium shedding light on the explanation of irreversible thermodynamics revolving around the second law. Finite temperature systems that are macroscopically large and at macroscopic times show a stochastic dynamics that is eventually controlled by the second law insisting that entropy has to increase in the future time direction. The insight is that this is just a consequence of the laws of quantum many body physics. The essence is that one can prepare a system in a pure state that is subjected to unitary time evolution. But in order for it to become observable one has to collapse the wavefunction and in a thermodynamically large quantum system such an observer will discern *the stochastic dynamics of a dissipative classical system.* The rate of entropy production is tied to the many body entanglement – the more strongly the states in the game are entangled, the faster it goes. This is an insightful affair with ties to quantum chaos and so forth. I recommend the reader to have a closer look at recent treatises [5].

Linear response is a simplifying circumstance: Chakraverty *et al.* [16] were the first to realize that the "heat production" in the quantum critical fluids submits to a principle of extraordinary simplicity which is at the same of an extraordinary weirdness within the reference frame of a "particle physicist". It is an instance where wording matters to help the human mind to get acquainted to the mathematical concept: I did something good for physics when I invented myself [4] the quote "Planckian dissipation."

I already discussed the way that this can be computed in principle using the thermal field theory machinery. Compute the stat. phys. correlation functions in Euclidean space time as these get modified by the time circle finite size scaling. Subsequently continue to Lorentzian time. But I already stressed the trouble that here the Euclidean "information loss" problem becomes insurmountable when analytical formulae are not available dealing with the regime where $\hbar\omega << k_B T$.

Given the Eigenstate thermalization principle the system has to behave as a classical, dissipative system at finite temperatures and macroscopic times. But such systems also give in to universal principle. We are typically interested in a *non-conserved* order parameter, associated with the spontaneous breaking of symmetry on the "ordered" side of the transition. Since such a field is not protected by a global conservation law is should be subjected to the phenomenon of *relaxation*. Having it switched on by applying an external order parameter stabilizing field (in infinitisemal form in linear response), by switching off this field the order parameter will *disappear*, relax on a characteristic time scale called the relaxation time. The work exerted by the external source will thereby turn into heat: this relaxation time is therefore linked generically to dissipation, the time it takes to generate entropy.

Such relaxational processes may be quite complicated, involving a hierarchy of relaxation times – see the discussion on the memory matrix in Section (VII). However, when it is governed by a single relaxation time it gets very easy. The EOM of the order parameter field will then be of the simple form,

$$\frac{d\Psi}{dt} + \frac{1}{\tau_\Psi}\Psi = M_\Psi \tag{25}$$

M_Ψ is the force exerted by the external source term, while τ_Ψ is the relaxation time. Inserting an oscillating source $M_\Psi(t) = M^0 e^{-i\omega t}$ it is an elementary exercise to derive the form of the dynamical susceptibility,

$$\chi_\Psi(\omega) = \chi_\Psi^0 \, \frac{i}{\frac{1}{\tau_\Psi} - i\omega} \tag{26}$$

the absorptive part describes a Lorentzian peak with width $1/\tau_\Psi$ centred at $\omega = 0$, This may be familiar from the textbook Drude theory, a typical example of a simple relaxation response (Section VII).

Back to the Euclidean field theory. We are interested in the behaviour of the Euclidean correlators at imaginary times long compared to the radius of the time circle, representing the $\hbar\omega << k_B T$ regime. As I already stressed, it is practically impossible to compute them. But now we can rely on simple reasoning: (a) assert that in this regime one is encountering classical, dissipative physics, (b) We observe that on the quantum side there is only *one* characteristic time that enters the ω/T cross over function: the radius of the time circle itself! Hence, the outcome should be that the classical relaxational dynamics can only know about a single time, and the high school theory Eq. (25) should therefore apply.

This has the unavoidable consequence that,

$$\tau_\Psi = A_\Psi R_\tau = A_\Psi \frac{\hbar}{k_B T} \tag{27}$$

where A_Ψ is a universal amplitude associated with the universality class and this a parametric factor that has to be of order unity. This incredible simple result is the Planckian dissipation.

The issue is that by importing Planck constant scales appear in the dimensional analysis that otherwise do not exist. The case in point are the UV Plack scales of quantum gravity, obtained by combining $\hbar$ with Newton's constant. The Rydberg scales of atomic and molecular physics are yet another example. But dissipation is associated with temperature, $k_B T$ having the dimension of energy. Planck's constant has the dimension of action which is energy times time. Henceforth, $\tau_\hbar = \hbar/(k_B T)$ is the characteristic time associated with temperature T. This time now becomes the time ruling the dissipative dynamics but only so dealing with a quantum system that is scale invariant (modulo the time circle) in Euclidean space time. Anybody can imagine something simpler than this wisdom?

On a side, a strange folklore appeared trying to argue this otherwise, even blaming my person for its origin. This departs from the *quantum mechanics* wisdom called the energy time uncertainty relation: $\Delta E \Delta t \geq \hbar$. Assert that $\Delta E = k_B T$ suggesting a characteristic time $t = \tau_\hbar$. But this is ludicrous, quantum mechanical systems of a few degrees of freedom to which this applies are of course not subjected to the genuine quantum field theory in the above! To simplify is good but do not take it too far: after all we have learned something since 1927.

I am also responsible for the conjecture that $\tau_\hbar$ is the shortest possible relaxation time in a linear response context, permissible by the laws of physics. This is entirely motivated by the Euclidean field theory reasoning in the above. To which degree it is absolutely certain that at times long compared to the Euclidean time circle one meets invariably classical dissipative physics, assumption (a) in the above? I am not aware of a mathematical proof but it is invariably the case in 1+1D integrable systems including the transversal field Ising model playing a key role in Sachdev's book. Better testimony follows from AdS/CFT dealing with field theories in the boundary that are a-priori not integrable where it is hardwired in the bulk geometry.

The other crucial condition is the scale invariance in Euclidean space time, assumption (b) in the above. This relies on generic behaviour associated with the finite size scaling in a scale invariant system – this may even not rely on the stoquastic nature of the problem where the rule is well established. When the system is characterized by internal scales as the stable SRE phases away from the critical point, these scales will interfere in the finite size scaling. Euclidean correlators are no longer only dependent on the radius of the time circle but also on the rigidity scale having the effect that relaxation is delayed: this is the reason that the Planckian dissipitation limit is *never* reached in particle physics.

A familiar example is the Fermi-liquid. In a way it is a most extreme example of a "classical" (SRE) system that is as quantum-mechanical as can be with the Fermi-energy representing a gigantic quantum zero point motion energy, see next section. The role of emerging conserved quantity is taken by *single particle* momentum, again next section. The relevant relaxation time is the time it takes for the finite temperature quasiparticles to collide: $\tau_{\text{col.}} \simeq E_F/(k_B T)\tau_\hbar$. This relaxation time is stretched by the factor $E_F/(k_B T)$ where E_F is the scale associated with the "rigidity" of the Fermi liquid.

The interest in Planckian dissipation is presently flourishing especially in the experimental condensed matter community. It appears to be quite ubitquitous in governing *transport* properties, with as icon the famous linear resisitivity in cuprate strange metals that is experimentally proven to be governed by $\tau_\hbar$. But this is actually a context where the Planckian dissipation conjecture does not apply directly. The difficulty is that this transport is controlled by a nearly conserved hydrodynamical soft mode, rooted in the *total momentum* being conserved in a translationally invariant system. The current relaxation can therefore not be universal in the Planckian sense: it eventually will reflect the way that the translational symmetry breaking survives in the deep IR. I will come back to this substance matter at length in Section (VII).

I already announced that I will ignore the non-equilibrium agenda, but let me shortly advertise here yet another development that is closely related to the Planckian dissipation theme [7]. In analogy with the Lyaponov exponents governing classical chaos one can define a quantum Lyaponov time. This arises in a non-equilibrium setting where one suddenly switches on a finite perturbation. One can then ask the question, how long does it take before the system has delocalized in the many body Hilbert space? It becomes then impossible to keep track of the further time evolution because of the exponential complexity that prohibits to deduce from VEV's where the system departed from. A lot more is needed than in linear response. Besides the quantum supremacy of the excited states and the absence of scale in the states taking part to the non-equilibrium time evolution one also needs all-to-all coupling in the Hamiltonian. This "scrambling time" can be computed conveniently using "out-of-time-order" (OTOC) correlators, and it has been mathematically demonstrated that this is bounded from the below by $\tau_\hbar$. Interestingly, this bound is saturated in the large N CFT's of holography.

D. Improvising quantum-critical metals: the Hertz-Millis model.

The take home message is that we understand so much of the stoquastic quantum critical state that a tight phenomenological framework is available to test it in experiment. This is all rooted in the powers of scaling; this is in close parallel to the history of the thermal critical state where Kadanoff developed the phenomenological scaling theory well before the more "microscopic" Wilson RG became available. The charm is just in the Wick rotation ploy where the familiar properties of the thermal critical state are twisted into the branch cuts, the quantum critical wedge and the Planckian dissipation.

The question that immediately arises, where is the experimental realization of such a strongly interacting stoquastic quantum critical state? One would like to find a no-frills version, in particular a sign-free spin system with an Euclidean version that is well charted in its thermal incarnation. Although experimentalists tried hard, no such case was until now identified.

One may have a second thought. Presently a grand pursuit is unfolding in the form of the creation of the quantum computer. The difficulty is in keeping this contraption isolated from the outside world to a degree that the coherence of states delocalized in the vast many body Hilbert space can be maintained. Right at the quantum critical "singularity" one is dealing with such states and it is as if it is impossible for nature itself to avoid circumstances that spoil this quantum supremacy. In fact, the same day that I was writing this passage the news broke that two groups managed to "program" a cold atom quantum simulator to deal with the transversal field Ising model – it is the closest approach I have seen coming out of experimental laboratories [20].

A case in point is the experimental realization of a literal transversal field Ising model. Ising spin systems are quite abundant, typically insulating salts containing rare earth elements. There is nothing easier than applying an external transversal magnetic field. Such a system was explored by Aeppli and Rosenbaum and coworkers in the late 1990's [21]. A complicating circumstance was encountered. Because of the dipolar interactions the classical (Ising) spin system is frustrated, exhibiting spin-glass properties. Glasses are generically characterized by extremely slow relaxation. The authors demonstrated that this speeds up significantly by the transversal field induced tunnelling in the complex energy landscape characteristic for glasses. This was actually the birth of quantum annealing, exploited among others by the d-wave quantum computer. Similarly, upon going to even lower temperatures one encounters the interactions with nuclear spins that will act as a heat bath decohering the electron spin system.

Nevertheless, quite a large number of systems were identified exhibiting quantum phase transitions but invariably these cannot be related straightforwardly to well understood thermal analogues. Invariably, these occur in metallic-like systems and the sign problem discussed in the next section is in the way of a rigorous understanding. Perhaps the best examples are the two dimensional systems exhibiting superconducting-insulator transitions that were conjectured to reflect the $U(1)$ (XY) universality class associated with the fluctuations of the superconducting phase as early as the late 1980's [22]. But these examples are further complicated by disorder, the role of fermion excitations as well as the fact that the experiments in first instance rely on transport measurements where conservation laws may obscure the quantum critical features. Another example are the "plateau transitions" observed in the quantum Hall effects [23]. These are obviously of the non-stoquastic kind (fermions and time reversal symmetry breaking) and are generally perceived as not understood.

Starting in the early 1990's a host of metallic quantum critical systems were discovered, the great majority of them in the context of the "heavy-fermion" lanthanide- and actinide intermetallic compounds [24]. These typically show magnetic order and applying influences like pressure or magnetic fields these can be tuned through a zero temperature transition where the magnetic order disappears at zero temperature. These exhibit the characteristic quantum critical wedges in the temperature-coupling constant plane, bordering a region characterized by anomalous transport properties including linear resistivities that appear to be reflect the Planckian dissipation. In addition, at low temperatures one finds often a superconducting "dome" centred at the quantum critical point.

The community standard is to view such transitions departing from the very early (1975) work by John Hertz, while Andy Millis ironed out some minor flaws more recently: the "Hertz-Millis" theory [25]. Hertz departed from a Fermi-gas subjected to weak interactions. These may then exhibit thermal transitions where either weak ferro- (Stoner-like) or anti-ferromagnetic order of the "nesting" spin density wave kind may appear. Asserting that the transition is mean field which is a good idea given the large BCS-style coherence lengths suppressing the fluctuations one can employ time dependent mean field ("RPA", "bubble sum") to obtain the relevant order parameter susceptibility describing the approach to the transition,

$$\chi_{\Psi,\Psi}(\omega, q, T) = \frac{\chi^0(\omega, q, T)}{1 - J_q \chi^0(\omega, q, T)} \tag{28}$$

where $\chi^0(\omega, q, T)$ is the Lindhard function of the Fermi gas (next section) and J_q the momentum dependent interaction strength.

The vanishing of the denominator than signals the onset of the order: $\chi^0(\omega = 0, q, T) = 1/J_q$. When this condition is reached as function of decreasing temperature at $q = 0$ ferromagnetism will set in, or either an antiferromagnetic spin density wave when it happens at a finite momentum q. I will show how this works in more detail in the discussion of holographic superconductivity (Section VI E 3): a peak start to develop in χ" in the normal state centred at an elevated energy having a comparable width. Upon lowering temperature this peak moves down and narrows to turn into a delta function at zero energy right at the quantum critical point.

Hertz addressed this as a zero temperature transition using the thermal field theory language that was in the 1970's not yet widely disseminated in the condensed matter community. One can recast it in the form of the order-parameter field effective action like Eq. (17), to add a Yukawa coupling to the Fermi gas of the form $\vec{\Psi} \cdot \psi^\dagger \vec{\sigma} \psi$ where $\vec{\sigma}$ are Pauli matrices and ψ the fermion field operators. For a non-conserved order parameter the Landau damping of the order parameter by the electron-hole excitations yields after integrating out the fermions using second order perturbation theory a modification of the kinetic term $\mathcal{L} \sim (|\omega| + q^2 + m^2)|\Psi(\omega, q)|^2$. This implies a "diffusion" dynamical critical exponent $z = 2$, as I already announced in section (III B 2). Given that the effective dimensionality of Euclidean space-time is $d + z$ for $d > 2$ one is above the upper critical dimension (typically $d_{u.c.} = 4$ for simple magnetic transitions) and the self interaction w can be ignored at the IR fixed point: these transitions are supposed to be of the non-interacting kind.

The difficulty is that "integrating out" requires that the fermions are fast as compared to the order parameter fluctuations. For any finite correlation length this is not quite true since particle-hole excitations occur at lower energies than the mass of the order parameter field. One has to address the "backreaction" of the critical order parameter fluctuations on the fermions. In higher dimensions it is argued that these are strongest in the pairing channel: the critical fluctuations act as an efficient pairing glue, driving a superconducting state [24]. The superconducting gap then protects the order parameter from potential 'danger' coming from massless fermions. Specifically in two space dimensions it was found that there are yet other, more subtle IR divergences associated with the fermions and an effort developed addressing this affair with quite sophisticated means [26]. To the best of my understanding this is still not quite settled.

A vast literature, both experimental- and theoretical, evolved [24]. Resting on the mean-field nature of Hertz-Millis many properties can be computed and it was argued that a subset of the Heavy-Fermion style quantum phase transitions appear to be consistent. However, also a group of "bad actors' were identified where this is not the case. This trouble is rooted in the fact that the physics of the metal is at the scales of interest not quite like a

Fermi-gas. Surely in the case of the heavy fermion "QCP's" the interactions in the UV are very strong, encapsulated by Anderson lattice models [24]: "Hubbard electrons" hybridizing with weakly interacting band structure state. Because of the fermion signs (next section) these escape a controlled mathematical description. Likely the metallic states are of the densely entangled "quantum supreme" kind but upon cooling down at low temperatures the heavy Fermi-liquids *emerge* as "instabilities" of this uncharted metallic stuff, as well as the magnetic order.

Recently a step in this direction was taken in the form of artificial fermion models characterized by a fine tuning making it possible to cancel the fermion signs (see e.g. [27]). The effective actions descending from such strongly interacting fermions are yet very different from the Fermi-gas kind and these can only be addressed by quantum Monte Carlo. Far from settled, this work does indicate that even in sign free models such fermionic QPT's behave differently. But nature does not give in naturally to sign cancellations and the general nature of such metallic quantum phase transition is shrouded behind the sign problem brick wall, see next section.

There is much more to be said regarding these metallic quantum critical points but perhaps surprisingly it is in the present context a side-line. We are heading towards what holography has to tell about quantum supreme matter. Contrary to some folklores, it has little to say about such quantum phase transitions. Within the established condensed matter paradigm such a transition is a necessary condition to avoid the SRE product states. But holography is insisting on the existence of "quantum critical phases", in essence densely entangled generalizations of the Fermi-liquid as I will highlight in Section (VI).

It became a community consensus in the cuprates that the strange metal physics around optimal doping also should be rooted in a QPT taking place right at optimal doping. At the time that we wrote the Nature "consensus document" [1] we also took this for granted. Accordingly, in the phase diagram Fig. (1) we indicated a quantum critical wedge – its fingerprint par excellence in experimental data, obviously with doping level as zero temperature tuning parameter. But as in the other cases one should be able to identify the order parameter that is disappearing. The experimental machinery of condensed matter physics is just tailored to detect order but despite an intense effort it could not be detected. The "intertwined" stripe etcetera orders appear to have vanished already at doping levels that are significantly lower than where the QPT should be located at a critical doping $p_c \simeq 0.19$.

The idea was born that some form of "hidden order" was in charge, a symmetry breaking that is difficult to detect with the experimental machinery.

One such form of hidden order was proposed that although still controversial deserves to be taken quite seriously: the spontaneous diamagnetic "loop currents" proposed by Varma ([28], Section VIII) This breaks time reversal invariance but not translations and is hard to detect – evidence was claimed for it in neutron scattering measurements but is not seen unambiguously in muon- and NMR measurements that should be exquisitely sensitive for such order. But in addition it has attractive ramifications such as the absence of a specific heat anomaly at the thermal transition, an induced pairing interaction favouring d-wave superconductivity and even a controversial claim that it could drive $z \to \infty$ local quantum criticality.

But this view changed drastically by the very recent experimental evidences for a highly anomalous *first order* like zero temperature phase transition residing at p_c. Similarly, in the last few years evidences have been accumulating that the *overdoped* metal realized at $p > p_c$ is a quite strange metallic phase as well. As I will discuss in Section (IX A) these are all evidences supporting the notion that both the under- and overdoped metals are in fact "quantum critical phases" that may well be governed on by general physics principle as suggested by holography.

IV. QUANTUM SUPREMACY AND THE (FERMION) SIGN PROBLEM.

At this instance we have arrived at a frontier of human knowledge. Our understanding of matter eventually hinges on the availability of mathematical machinery that is a prerequisite for this understanding. We learned that the material in the textbooks of physics hinges on the existence of the SRE product vacuum states. In the previous section we learned that by resting on the machinery of statistical physics we can extend this to the strongly interacting quantum critical point, but this requires that the problem is stoquastic.

But stoquastic systems are in fact very rare in nature – only real boson systems like ^{4}He satisfy the conditions without fine tunings of various kinds. *Generic finite density quantum systems are characterized by Euclidean path integrals that are not of the statistical physics kind:* nature is intrinsically non-stoquastic.

The bottom line is that no mathematical machinery is available to tackle such problems.

It is of course not pleasant for theoretical physicists to admit that their hands are empty, and accordingly in the mathematically inclined part of the community this was just brutally worked under the rug. In such a sociological circumstance it is typically beneficial to refer to quotes in a remote past by the prophet Feynman. In fact, in the illustrious Feynman-Hibbs path integral book [29] you will find reference to the failure as perceived by Feynman of the path integral to shed light on fermion problems.

Until rather recently, the (fermion) sign problem was often put away as a software engineering problem faced by computational people, not of interest to serious theoretical physicists. It is still a prevalent attitude in the string theory community, annoyingly so because it is quite obvious that the charm of AdS/CFT is in that it makes possible to have a look on what is going on behind the "fermion sign brick wall." The best therapy is to interact with the brave part of the computational community that has been resiliently chasing the sign problem for the last thirty years or so. Computers keep us honest for the simple reason that these do not offer the opportunity to look away from the trouble.

This section is intended to offer a pedestrian introduction in the sign problem, to get some intuition of where the trouble is. All we can do is to diagnose the problem, as an introduction to the potential cure offered by holography. I will first review a story that is for no good reason not found in the quantum field theory textbooks: the way that fermion- and boson statistics is encoded in the first quantized (worldline) path integral (section IV A). This yields a basic insight in why fermion problems are so much harder than their bosonic, stoquastic counterparts. I will then step back to focus in on the state of fermions that is understood: the Fermi-liquid, being a "classical" (i.e. SRE product) state in disguise with a physics that is however singularly different from anything that can accomplished by bosons (section IV B). Section IV C is the crucial where I will explain a mathematical theorem stating that non-stoquastic problems are in general of exponential complexity: "NP-hard".

I will then step back to physics and introduce the "Mottness" condition, being a common denominator of the strongly correlated electron systems forming the condensed matter theatre where all of it takes place (section IV D). I will highlight here yet another unfamiliar story that allows us to diagnose the nature of the horrid sign problem as encountered in this specific context that is invariably at work in the condensed matter mystery systems. For completeness I will finish this section with a very concise summary of the recent progress in the computational community.

A. Exercising the fermion signs: the path integral of the Bose and Fermi gas.

In order to acquire intuition regarding the way that the sign problem in the path integral relates to the Pauli principle of the canonical formalism. It is informative and entertaining to consider the simplest of all problems: the non-interacting Fermi- and Bose gas in the first quantized (worldline) path integral formalism. This should be part of the basic training of any physicist but it seems that for accidental historical reasons it did not enter the textbooks. An exception is found in the book "Path Integrals" by Kleinert [30] and let me outline the way it works.

The worldline representation is a pleasant arena to familiarize oneself further with the Euclidean path integral that we already encountered in the previous section. Let us first exercise this with the elementary problem of a single free quantum mechanical particle in first quantized representation. Let us depart from the "transition amplitude", the probability that a particle with mass M in D space dimensions that is created at x_a, t_a will arrive at the space-time coordinate x_b, t_b as explained in elementary texts,

$$\langle x_b t_b | x_a t_a \rangle = \frac{1}{\sqrt{2\pi i \hbar (t_b - t_a)/M}^D} e^{\frac{i}{\hbar} \frac{M}{2} \frac{(x^a - x_b)^2}{t_a - t_b}} \tag{29}$$

Upon Wick rotation to imaginary time τ,

$$\langle x_b \tau_b | x_a \tau_a \rangle = \frac{1}{\sqrt{2\pi \hbar (\tau_b - \tau_a)/M}^D} e^{-\frac{M}{2\hbar} \frac{(x^a - x_b)^2}{\tau_a - \tau_b}} \tag{30}$$

Here one already infers the magic: this is identical to the two point function of a "polymer", an elastic line, subjected to Gaussian thermal fluctuations after associating the line tension with M and temperature with $\hbar$. At finite physical temperature the imaginary time direction turns into the circle and all that can happen in the vacuum state is that the "polymer" forms a closed loop "lassoing" the time circle. The quantum partition sum is then determined by the effective classical partition sum of this "ring polymer" (see underneath). An elementary computation yields,

$$Z_{\text{part}} = \frac{V_D}{\sqrt{(2\pi \hbar \tau_\hbar)/M}^D} \tag{31}$$

which you may recognize as the partition sum of a single quantum mechanical particle in the spatial continuum.

Let us now consider an ensemble of such particles. We have to add the postulate that particles are either fermions of bosons characterized by (anti-) symmetrized Fock space. Depart from the first quantized configuration space where $\mathbf{R}(\tau)$ specifies the coordinates of all particles at (imaginary) time τ, $\mathbf{R} = (\mathbf{r}_1, \dots, \mathbf{r}_N) \in \mathbb{R}^{Nd}$. The partition sum can be expressed as the integral over this configuration of the diagonal density matrix evaluated at $\tau_\hbar$, the wisdom we saw already at work in the previous paragraph,

$$\mathcal{Z} = \mathrm{Tr} e^{-\beta H} = \int \mathrm{d}\mathbf{R} \rho(\mathbf{R}, \mathbf{R}; \beta). \tag{32}$$

But dealing with indistinguishable fermions or bosons one has to (anti) symmetrize the wave function and in this worldline path integral representation this translates into summing over all $N!$ *permutations* $\mathcal{P}$ of the particle coordinates,

$$\rho_{B/F}(\mathbf{R}, \mathbf{R}; \beta) = \frac{1}{N!} \sum_{\mathcal{P}} (\pm 1)^p \rho_D(\mathbf{R}, \mathcal{P}\mathbf{R}; \beta), \tag{33}$$

where

$$\rho_D(\mathbf{R}, \mathbf{R}'; \beta) = \int_{\mathbf{R} \to \mathbf{R}'} \mathcal{D}\mathbf{R} \exp(-\mathcal{S}[\mathbf{R}]/\hbar), \tag{34a}$$

$$\mathcal{S}[\mathbf{R}] = \int_0^{\hbar\beta} d\tau \left(\frac{M}{2} \dot{\mathbf{R}}^2(\tau) + V(\mathbf{R}(\tau)) \right), \tag{34b}$$

$V(\mathbf{R}(\tau))$ is the potential energy which can be due to either interactions of external potentials that we will ignore for the time being. Remarkably, the quantum statistical requirement turns into a topological affair. The way this works is that one departs from a bunch of worldlines at imaginary time τ_0. One follows the "word history" by evolving along the imaginary time direction until one has traversed the time circle arriving again at τ_0: at this instance one has to connect the worldlines again but the (anti) symmetrization requirement insists that this can be accomplished a-priori in any way. Dealing with N particles this can be accomplished by having N single particle "loops". But one can also wrap one worldline N times around the time circle, see Fig. (7). For the latter case, on a single time slice one discerns N particles, worldlines piercing through through, but these share all *one worldline*. This is the origin of the Feynman quote that there is "only one electron in the universe".

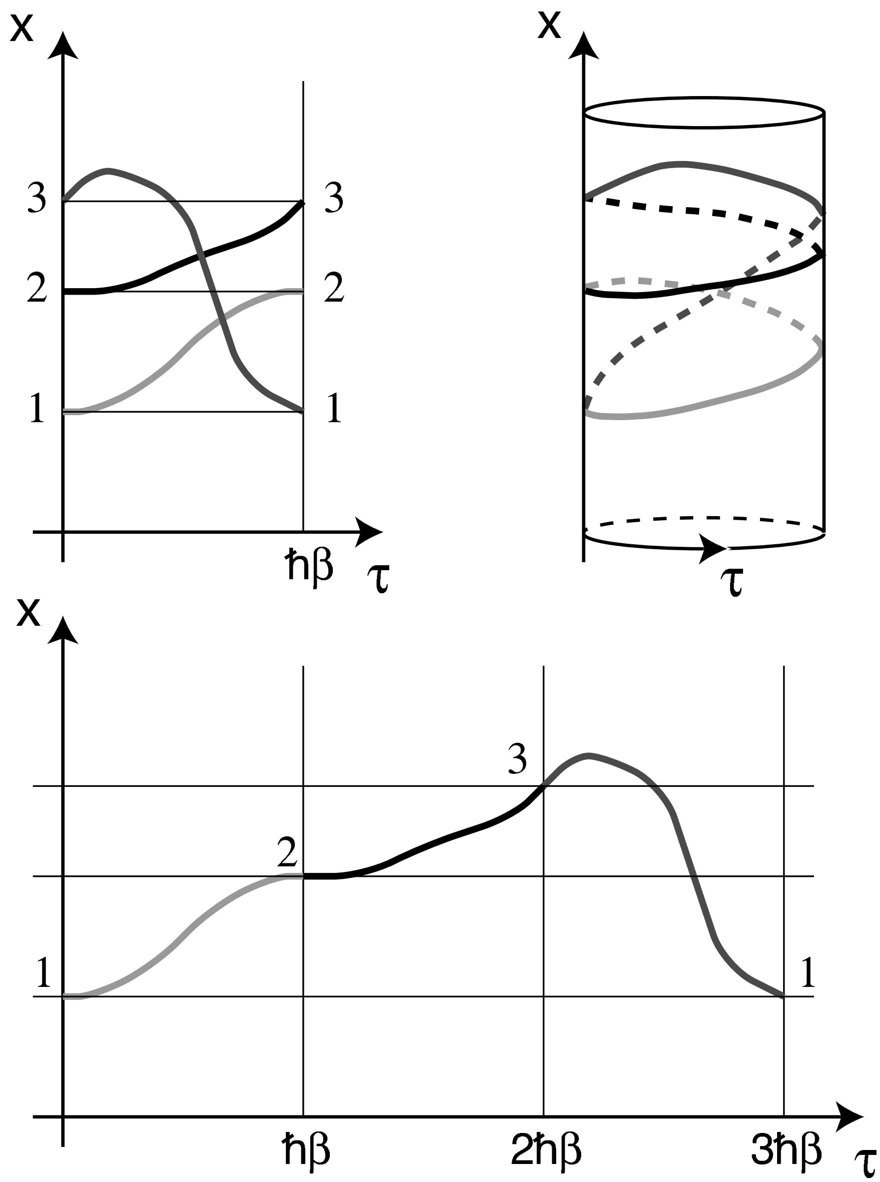

FIG. 7. In Euclidean signature worldlines are wrapping around the imaginary time circle, see main text.

Let's inspect how this works as function of temperature. The worldlines will meander along the imaginary time direction like polymers with fixed string tension $\sim M/\hbar$. When temperature is high the time circle is short and upon traversing the time circle the net average distance that is traversed will be small with the effect that every particle line connects with itself. This is the path integral way to understand that above the degeneracy temperature the system becomes a classical gas. However, upon increasing the time circle upon lowering temperature this meandering distance becomes of order of the interparticle distance: the de

Broglie thermal wavelength becomes of order of the interparticle distance and degeneracy sets in. Consider the probability in the case of two particles that a single worldline wraps twice around the time circle : this may become of the same order as two individual "lasso's". But now the difference between fermions and bosons becomes also consequential through the $(\pm 1)^p$ factor in Eq. (33). For bosons both the single- and two wordlines contribute with a probability factor > 0 to the quantum partition sum. However, for the fermions the two single loops have a positive weight but the one wordline wrapping twice around the time circle carries a "negative probability": you are facing the sign problem!

The next step is a bit of an intricate combinatorics exercise [30**?**] that only works for the free problem. The outcome is that one can write the partition sum in terms of the *winding number* w. One can just rewrite it as a number over worldlines winding w times around the time circle. The grand canonical partition sum becomes,

$$\begin{aligned} Z_G(\beta,\mu) &= \sum_{N=0}^{\infty} Z_{B/F}^{(N)}(\beta) e^{\beta\mu N} \\ &= \exp\left(\sum_{w=1}^{\infty} (\pm 1)^{w-1} \frac{Z_0(w\beta)}{w} e^{w\beta\mu} \right), \end{aligned} \tag{35}$$

corresponding to a grand-canonical free energy

$$\begin{aligned} F_G(\beta) &= -\frac{1}{\beta} \ln Z_G(\beta,\mu) \\ &= -\frac{1}{\beta} \sum_{w=1}^{\infty} (\pm 1)^{w-1} \frac{Z_0(w\beta)}{w} e^{\beta w\mu}, \end{aligned} \tag{36}$$

with the $\pm$ inside the sum referring to bosons (+) and fermions (−), respectively. This is quite elegant: one just sums over worldlines that wind w times around the time axis; the cycle combinatorics just adds a factor $1/w$ while $Z_0(w\beta)\exp(\beta w\mu)$ refers to the return probability of a single worldline of overall length $w\beta$. This further simplifies in the spatial continuum to,

$$\begin{aligned} Z_0(w\beta) &= \frac{V^d}{\sqrt{2\pi\hbar^2 w\beta/M}^d} \\ &= Z_0(\beta)\frac{1}{w^{d/2}}, \end{aligned} \tag{37}$$

Yielding the free energy and average particle number N_G, respectively,

$$F_G = -\frac{Z_0(\beta)}{\beta}\sum_{w=1}^{\infty}(\pm 1)^{w-1}\frac{e^{\beta w\mu}}{w^{d/2+1}}, \tag{38a}$$

$$N_G = -\frac{\partial F_G}{\partial \mu} = Z_0(\beta)\sum_{w=1}^{\infty}(\pm 1)^{w-1}\frac{e^{\beta w\mu}}{w^{d/2}}. \tag{38b}$$

Finally, the sums over windings can be written as,

$$\sum_{w=1}^{\infty}(\pm 1)^{w-1}\frac{e^{\beta w\mu}}{w^{\nu}} = \frac{1}{\Gamma(\nu)}\int_0^{\infty} d\varepsilon \frac{\varepsilon^{\nu-1}}{e^{\beta(\varepsilon-\mu)} \mp 1}, \tag{39}$$

and we recognize the textbook expression involving an integral of the density of states ($N(\varepsilon) \sim \varepsilon^{d/2}$ in d space dimensions) weighted by Bose-Einstein or Fermi-Dirac factors.

I presume the reader has captured the physics entertainment that is going on in the above. The indistinguishability of the quantum particles translates into the "wind around the time axis as often as is allowed" – this is a topological affair, associated with "lassoing the cylinder" (homotopy group $\Pi_1(S_1) = Z$). By subtle combinatorics this then eventually translates in the texbook Bose-Einstein and Fermi-Dirac distributions familiar from the canonical formalism. The big deal is that one immediately infers that for the bosons the sum over the winding numbers is also a stochastic affair – every winding configuration is characterized by a Boltzmann weight probability, and this sums up to the Bose-Einstein distribution. The problem can actually as wel be interpreted in terms of the classical problem of "ring polymers" winding around a cylinder in space.

But how about the fermions? In the "winding sum" Eq. (39) one observes the $(-1)^{w-1}$ factor: uneven windings contribute with a positive "Boltzmann weight", but even windings are like "negative probabilities". The overall contributions of adjacent w's are nearly the same when w gets large and these nearly cancel each other. The bottomline is that the free energy is pushed upward, and these "destructive interferences" are the origin of the Fermi-energy – the fermion system at finite density is characterized by an enormous zero point motion energy.

Dealing with free fermions this alternating sum just turns into the Fermi-Dirac distribution but when interactions get important this sum becomes ill defined. This is the first quantized way to appreciate the origin of the sign problem.

To appreciate this a bit more, let's focus in on a context where the first quantized representation is the natural one: the Helium quantum liquids. At $\simeq 3$ K both the fermionic ^{3}He and the bosonic ^4He form a dense *classical* van der Waals liquid. The description of such fluids has been a central challenge in the classical fluid community. One can picture it as a form of dense traffic. The He atoms are in first instance like hard, impenetrable balls that occur at a high density so that they all touch each other, and this cohesive fluid is kept together by the weak, long range van der Waals attractions. The motions are like stop and go traffic, it is extremely collective. For your car to move, a mile away or so a car moved and a string of cars followed to eventually make it possible for you to get your foot from the brake pedal. The classical van der Waals fluid was tackled by brute force numerics, based on solving Newton's equations of motions with added noise terms – "molecular dynamics".

But one now lowers temperature further so that quantum degeneracy becomes on the foreground. Already in the 1950's or so the liquid form factor was measured by neutron scattering showing that down to the lowest temperatures the local physics does not change at all: microscopically it is in both isotopes the same van der Waals traffic jam, eventually kept fluid by quantum zero point motion.

This sounds like an insurmountable problem: the UV "coupling constant" is in a way close to infinite given the hard steric repulsions. But nevertheless the bosonic ^{4}He problem can be regarded as completely charted theoretically at least for equilibrium properties. The reason is that the worldline quantum Monte-Carlo just gets it done as demonstrated by David Ceperley in the 1990's as one of the great triumphs of QMC [32].

How does this work? Let us first revisit the free bosons: we know from the canonical formalism that at some temperature Bose-Einstein condensation will take over, where a finite density of Bosons will occupy the $k = 0$ single particle momentum states. The way this works in the worldline representation is that by lowering temperature at average the number of windings will increase. At some temperature worldlines with $w \to \infty$ will acquire a finite probability and this corresponds with the Bose-Einstein condensation temperature. In fact, at $T = 0$ the probability to find a particular winding $P(w)$ becomes completely w independent. The value of $P(w)$ at $w \to \infty$ is associated with the superfluid density ρ_{SF} and the flat distribution means that all particles are in the condensate, $\rho_{SF} = 1$. This is in fact an example of weak-strong duality that I touched upon in Section (II). The winding of the worldlines in the path integral is actually encoding for the many-body entanglement and a flat winding distribution implies a form of maximal entanglement in position (number) representation. But in momentum (phase) representation it is just a simple tensor product.

Let us now consider the van der Waals fluid version. Because of the hindrance coming from the steric repulsions, the contribution from long windings are suppressed relative to the short windings. But given that eventually an infinitely long worldline can get through there will be a temperature where as for the free system P becomes finite at infinite w: this is the transition to "Helium II", the superfluid! The bottom line is that at zero temperature $P(w)$ is no longer w independent as in the free Bose gas: it is skewed towards small w but above a critical w_{crit} it will again become w independent although at a lower value than in the free system. This expresses that the zero temperature superfluid density is reduced significantly from the Bose-Einstein case, one of the well known properties of Helium. Surely, every equilibrium property can be computed, being right on top of experiment as Ceperly showed.

But let us now turn to ^{3}He, the fermionic version – a classic example of the difficulties associated with the "non-stoquastic" quantum physics. Departing from the van der Waals liquid conditions in the UV, it is obvious that the winding summation with the alternating signs turning into the Fermi-Dirac distribution is no longer working. What happens instead? The answer is an empiricism that added to the fame of Landau. He realized that at temperatures well below 1K all experimental properties are consistent with the emergence of a Fermi-gas formed from renormalized quasiparticles. In turn he formulated the Fermi-liquid theory to describe these renormalizations, in fact relying on (time dependent) mean-field theory that is in a modern (post RG) language validated by the local stability of the fixed point, see underneath.

One has to realize what is going on here. One departs also in the case of ^{3}He from the UV "quantized traffic jam physics" of the dense van der Waals liquid as proven by the liquid form factor measurements. Upon renormalization towards the IR a miracle happens: the impenetrable hard balls of the UV turn into perfectly non-interacting quasi-^{3}He particles that have just increased their mass by a factor of 10 or so (Fermi-liquid phenomenology) communicating only by the Pauli-principle! It is one of the dirty secrets of physics that is so painful that it got worked under the rug completely: nobody has even the faintest clue how this can happen!

To further demotivate wishful theoretical dreaming, as for ^{4}He one can unleash all the supercomputing power available to attempt to compute this brute force. But one then finds that when temperature gets well below the bare Fermi-energy the required computational sources starts to grow exponentially – this exponential complexity rooted in quantum supremacy appears to be fundamental: a no-go theorem has been claimed, see next section.

The take home message is that experiments show that the Fermi-liquid fixed point is amazingly stable: the ^{3}He example was later followed by a zoo of "heavy Fermi liquids" in electron systems characterized by up to a 1000 fold quasiparticle mass enhancements [24]. But the explanation of this excessive stability is completely in the dark: it is shrouded by the sign problem! This is the problem I alluded to discussing the metallic quantum phase transitions in Section (III D). Eventually very heavy Fermi-liquids are realized at very low temperatures surrounding the quantum critical point in the heavy fermion systems. But the UV is similar as in Helium, with interactions that are much larger than the bare Fermi-energy. By miracle somehow this manages to turn at very low temperatures in a Fermi-liquid but actually nobody can tell why this seemingly unreasonable feat is accomplished. I already emphasized that the Hertz-Millis "picture" departs from a free Fermi-gas in the UV that is *weakly* coupled to the fluctuating order parameter. That this became influential is just by the "mechanism of credibility by repeated publication". It was a great source of employment for theoreticians since many variations can be figured out on the basic theme resulting in thousands of papers.

B. The remarkable Fermi liquid.

As I already emphasized all we know for sure is that the Fermi-liquid is a "classical state of matter", in the sense that the vacuum is a short ranged entangled product state "spiced up" by the Pauli principle. Given that you are now used to first quantized path integrals this is a proper instance in this development to sketch the precision argument demonstrating that the Fermi-gas is devoid of entanglement. I already emphasized the crucial feature of entanglement that it should be representation independent – when one can find a representation where the state is an (SRE) product the state is not "quantum supreme" regardless what may be suggested by in this regard "awkward" representations. For two qubits this is easy – the bipartite von Neumann entanglement entropy yields a precision answer. In the many body context no such powerful device is available. I discussed already a couple of times the typical outcome that a state may appear to be maximally entangled being however an ESR product state in dual representation. How can one be sure that such a dual is not identified?

1. *The Fermi-gas is not entangled: the nodal surface representation.*

The precision argument identifying the Fermi gas as being strictly unentangled involves a less well known representation of the fermionic path integral. Ceperly discovered that one can re-shuffle the sign problem in the first quantized fermion path-integral into an object called the "nodal surface": this is defined as the hypersurface where the full dynamical density matrix is vanishing, closely related to the nodes of the Slater determinant in configuration space. It is a theorem that given full knowledge of this nodal surface one can reformulate the path integral in a stoquastic form: the nodes turn into reflective boundary conditions "trapping" the bosonic world histories.

Sergei Mukhin [**?**] realized that this becomes a surprisingly simple affair dealing with the Fermi gas in momentum representation where the nodal surface is equivalent to a *classical* "Mott insulator" in momentum space. The partition sum turns out to be identical to a system of classical particles with a hard core condition such that only one particle can reside at one momentum space "site". In turn these "classical particles in the momentum space optical lattice" live in a harmonic potential and the vacuum state is obtained by "filling the cup" with the Fermi surface corresponding with the brim, see Fig. (8). This shows once and for all that the free fermion Pauli principle is no more than a "classical information resource".

When we get to the core of these lecture notes in Section (VI), I will argue vigorously that the densely entangled holographic strange metals should be viewed as *generalizations* of the Fermi-liquid. It defines a template for the *gross* organization of these states, in parallel with the free Landau mean field theory above the upper critical dimension forms a template for the strongly interacting critical state. This defines a general *scaling* phenomenology that is "deformed" by anomalous dimensions that departs from the "covariant" scaling of the Fermi-liquid itself as identified in Section (VI D 3).

The key is that the Fermi-liquid is a non entangled "classical" state of matter – it is a SRE product vacuum. it is also a highly collective affair governed by strong emergence that only becomes precise in the thermodynamic limit. In this regard it should be viewed as

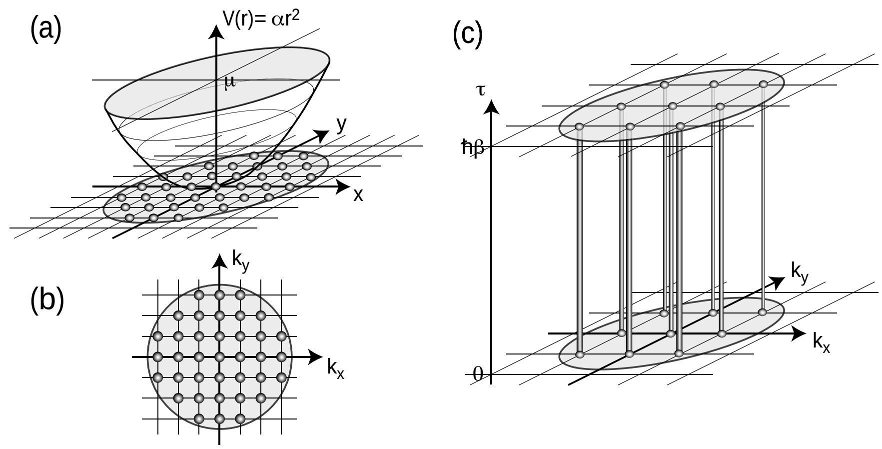

FIG. 8. Using Ceperley's nodal surface representation, the Fermi-gas maps on the problem of classical hard-core particles residing in a momentum-space "optical lattice" being trapped in a "harmonic crucible".

an analogue (perhaps even better, generalization) of the ordered states of "bosonic" matter characterized by an order parameter describing spontaneous symmetry breaking. But this acquires a fermionic twist: in fact, the *Fermi surface* takes the role of order parameter.

The easiest way to get a handle on this "order parameter" claim is by the realization that the Fermi-surface is a geometrical object ("surface") with a precise locus in the space formed by *single particle momentum* quantum numbers. But according to the *exact* quantum rules this space does not exists when the fermions submit to any form of physical interactions! This is simple: the single particle momentum operator $n_k = c^{\dagger}_k c_k$ does not commute with the interactions generically of form $\sim \sum_{k,k',q} V(\cdots) c^{\dagger}_k c_{k+q} c^{\dagger}_{k'} c_{k'-q}$. Total momentum is conserved in a homogeneous background but single particle momentum is not and therefore it is not quantized. But this is the same phenomenon that is underpinning the most familiar of all forms of conventional spontaneous symmetry breaking: the single particle position operator is not conserved either but in a crystal it *emerges* as a constant of motion: we know where the atoms are in space. This of course ascends from the fact that the crystal vacuum is an SRE product controlled by stitching together atoms in real space wave packets. This role is now taken by the unentangled Fermi gas.

The Goldstone bosons of conventional order are eventually rooted in the fact that the ordered system carries excitations that are driven by symmetry restoration. In this regard the "Fermi-surface order" is much richer. The "fluctuation of the Fermi surface" is in first instance embodied by the particle-hole excitations spanning up the "incoherent" Lindhard continuum. This in turn enriches the perturbation theory required to "dress" the product state with the short ranged entanglements using the modern quantum information language. Although one is implicitly after the stability of the "symmetry-breaking" of the Fermi-liquid, in the odd fashion explained in the above, this may be easily confused with plain-vanilla perturbation theory as has been a widespread misconception. Dealing with stoquastic order one focusses on the physics responsible for the fluctuations of the order parameter. The Fermi-liquid is unique in the regard that this factor is coincident with the *electron-electron interactions*. The reference 'product state' state associated with the "symmetry breaking" of the Fermi-liquid is the Fermi gas and the electron-electron interactions are responsible for the "fluctuations of the Fermi-surface."

The advantage is that one has a simple small parameter: the ratio of the interaction strength to the Fermi-energy. This is the central wheel of the perturbative weak coupling expansion around the Fermi-gas taking the form of the diagram "gymnastics" embodied by "AGD" [33]. Rooted in the intricate nature of the Fermi-gas excitation spectrum (the Lindhard continuum) one needs here the fanciful counting devices developed for QED – in fact, this kept the "diagrams" alive in the 1960's when the high energy physicists were temporarily looking elsewhere.

This came to full fruition relatively recently by the realization that this should be addresses in terms of the *functional* renormalization group by Shankar and Polchinski (see [34]). At every point on the Fermi-surface one can define a bunch of coupling constants and these all run a-priori in their own ways under renormalization, one needs *functions* of coupling constants. The bottom line is that when one departs from a *UV Fermi gas* switching on *small UV interactions* one can rigorously demonstrate that these are all UV irrelevant except for the Cooper channel. The only threat for this Fermi-liquid is that it will turn into a BCS superconductor in the presence of an attractive interaction.

But this does not shed *any* light on the ^{3}He "fermionic van der Waals liquid" puzzle, and equivalent problems in electron system (see underneath) that I just alluded to. It is absurd to assert that one can identify a Fermi-gas as the object controlling the van der Waals UV physics subjected to small interactions. All that one learns is that at the instance that the Fermi-surface starts to appear in a renormalization flow that is otherwise completely in the dark (the "hard Helium balls becoming transparent") it is like a funnel: upon further downward renormalization it is taking over the flow with the result that in the deep IR a perfect effective quasiparticle Fermi-gas is formed.

2. Is the Fermi-liquid critical or stable?

The upshot of the modern functional RG view on the Fermi-liquid is that one relies on the same *continuous* RG procedure as for the critical state. But it is yet physically quite different. The Fermi-liquid has an a-priori existence as a stable *state of matter*. When the fundamental pairing instability is avoided as it is in elementary metals like copper it persists all the way to zero temperature without invoking fine tuning of any kind. But at the same time all the IR physical responses of the Fermi liquid are powerlaws! This is actually the necessary condition for the success of the continuous functional RG.

Is the Fermi-liquid a stable- or critical state of matter? This question tends to cause confusion. You will learn to appreciate a main triumph of holography in the form of the "geometrization" of the RG flow in terms of the gravitational geometry of the bulk dual in the next sections. This reveals that a different principle is at work in the "geometry" of the RG flow. At a critical point it is controlled by *invariance* under scale transformation while the Fermi-liquid type of RG is controlled by *covariance* under these scale transformations. Keeping this as governing principle the holographic strange metals can then be considered as generalizations of the Fermi liquid.

For these future purposes let me list here some of the well known scaling properties of the Fermi-liquid. First, transport is tricky because of the hydrodynamical protection. Underexposed in elementary courses is the *zero sound* of the Fermi-liquid. Also at zero temperature total momentum is conserved. The Fermi-gas is of course a theoretical abstraction and in the real Fermi-liquid one has to account for short range repulsive interactions. These can be handled "at the fixed point" using RPA/time dependent "classical" mean field (the secret of Landau's Fermi-liquid phenomenology) and the outcome is that an "antibound" propagating mode "splits off" from the (Lindhard) particle-hole continuum at zero temperature. The physical interpretation is that the Fermi-surface "hardens", supporting an elastic response with regard to a "breathing" (s-wave) oscillation of the Fermi-sphere. This also modulates the area enclosed by the Fermi-surface in momentum space and it therefore corresponds to a density fluctuation: it is a sound wave. It is different from the finite temperature "first" sound associated with normal hydrodynamics, as strikingly observed in the form of the sound attenuation maximum in ^{3}He. In an electrically charged fluid this sound mode is promoted to the plasmon. The take home message is that zero sound is also generic in the holographic strange metals.

Such sound modes are massless and in a way also "algebraic" but this is not alluding to the "criticality" of the Fermi-liquid. Instead, this is rooted in the Lindhard spectrum: the particle-hole excitations around the Fermi-surface. These may be viewed with a statistical physics eye as the excitations around the "rim" of the classical-atoms-in-a-"harmonic" optical lattice of Fig. (8), which is of course precisely the same as it is presented in elementary quantum statistical physics courses.

The entropy (specific heat) is a first quantity that is rooted in the particle-hole exciations. It is of the Sommerfeld form, $S \sim T$ regardless dimensionality. The quasiparticles are effectively massless Lorentz invariant (linear dispersion relations near E_F) while their inverse lifetime $\sim \omega^2$ giving in to ω/T scaling that we learned to appreciate as reflecting scale invariance in Euclidean space time. The Lindhard function that is counting the "incoherent" electron hole excitations is associated with the spectral function of charge excitations – the imaginary part of the dynamical charge susceptibility $\chi"_\rho(q,\omega)$ – and at finite momentum it shows ω/T scaling behaving as ω^2 for $\omega >> k_B T$. The most famous example is the central wheel of the Bardeen-Cooper-Schrieffer theory of superconductivity. This is associated with the *pair* susceptibility $\chi_p(q,\omega)$. There is no global conserved quantity associated with the Cooper pairs and for the free Fermi gas it exhbits a *marginal* scaling dimension: $\text{Im}\chi^0_p(q = 0,\omega) \sim \omega^0$. It follows from Kramers-Kronig that the real part becomes logarithmically divergent, $\text{Re}\chi^0_p(q=0,\omega) \sim \ln(\omega)$. The mean field condition to find the critical temperature is $1/V = \text{Re}\chi^0_p(q = 0, \omega = 0, T = T_c) = N_0 \ln(T_{UV}/T_c)$, where V, N_0, T_{UV} are the strength of the attractive interaction, N_0 the density of states and T_{UV} the UV cut-off (e.g. phonon frequency), respectively. Actually, all of it submits to the energy-temperature scaling that we learned to appreciate in section (III) as a fingerprint of the quantum phase transition criticality.

In Section (VI) the "organizing principle" behind these power laws will be highlighted, making it possible to generalize them to genuine non-Fermi-liquids.

C. The Troyer-Wiese no-go theorem.

Computational complexity theory is a corner stone of mathematical computer science. It departs from the universal classical computer – the Turing machine – and asks the question how the amount of computer time scales with the size of the problem. Remarkably, it is then possible to identify various complexity classes and decide at least in principle in which class a particular algorithmic problems belongs. Consider a problem characterized by N bits of information; this is considered as benign when it can be solved in a time that scales polynomially with N (computer time $t \sim N^\alpha$) defining the class "**P**". However, there is also the class of "non-polynomial" ("**NP**") problems that are typically associated with an exponential growth of the computational effort, the "exponential complexity" $t \sim e^N$. Obviously, for large N this amounts to a no-go theorem. As an added subtlety, one can identify a subclass of **NP** problems characterized by the property that when an algorithm would be found that can solve it in polynomial time this can be used to solve *all* **NP** problems: the **NP**-hard class. This is obviously the holy grail of computation: when such an algorithm would be found one could dream that e.g. chaotic problems like weather forecasting could be extended over much larger periods.

Although often not emphasized, this also encapsulates the glass ceiling of the "unreasonable effectiveness of mathematics in natural science". Scribbling mathematical equations and solutions on a piece of paper is also a form of computation. When this works one is obviously dealing with problems that viewed from the computational complexity angle represents the *simplest* problems altogether. In the daily practice, the effective/phenomenological theories of physics are typically of the kind but in order to get the numbers right one has to mobilize a "polynomial effort" of a computer. To say it in a more provocative way, physics has been a cherry picking affair, filtering out those problems that are of polynomial complexity that can be cracked by mathematics. The vast stretches of reality that are of exponential complexity are left to the other sciences which are inherently entirely empirical with theories that are to physicists standards no more than heuristic hand waives. This is eloquently captured by the classic wisdom among young male physicists that "girls are way more complicated than quarks".

Troyer and Wiese presented in 2004 the remarkable claim [6] that the *equilibrium problem of strongly interacting fermions at finite density is* **NP***-hard.*. This departs from the now familiar territory that a-priori the Hilbert space of the quantum many body system is exponentially large. For the sake of the argument consider N qubits (or spins, whatsoever) and I already emphasized that the Hilbert space is spanned up by 2^N tensor products/classical bit strings labelled by c. The computation of the equilibrium quantum partition sum takes in path-integral representation the form of a Boltzmann partition sum modified by the possible occurrence of the "negative probabilities"

$$Z = \sum_c s(c)|p_c| \tag{40}$$

where $s(c) = \text{sign}(p_c)$ while $|p_c|$ takes the form of the Boltzmann weight $p_c = e^{-\beta E_c}$ of the equivalent classical stat. phys. problem arising in the Euclidean space-time of the path integral.

The generic polynomial algorithm to solve such problems is Metropolis Monte-Carlo. Although the configuration space is exponentially large, Metropolis sorts out the polynomially small sub-manifold where the solution of the problem resides entirely. When $s(c)$ is positive definite this always works with as exceptions the "strongly interacting" critical states and special "glassy" frustration problems. You are now of course used to the notion that Metropolis works dealing with ESR products in the canonical representation.

But this changes radically when $s(c)$ takes both positive and negative values according to Troyer and Wiese. Their argument is in fact very simple. The consider the following problem as being fully representative for the class of sign-full quantum problems,

$$H = -\sum_{\langle j,k \rangle} J_{jk} \sigma^x_j \sigma^x_k \tag{41}$$

where σ^x_j are the usual Pauli matrices, while the nearest-neighbour couplings J_{jk} take values randomly from 0 and $\pm J$. Evaluate this for a z-quantization and the random couplings correspond with random signs of the off diagonal matrix elements, which in turn translate into a "spin worldline" path-integral with random signs.

Proving the **NP** hardness is now trivial: in x quantization Eq. (41) is the Ising spin-glass problem, one of the frustrated examples that is well established to be **NP**-hard but only so when J takes positive *and* negative values– it is implied that therefore also the generic "sign-full" quantum problem is of exponential complexity.

D. Mottness as the sign problem amplifier.

In the spatial continuum electrons exclusively interact via the Coulomb interactions. Its long range nature is yet another, independent factor that adds to the stability of the Fermi-liquid. This is in essence not different from the wisdom in stoquastic systems that long range interactions are less "dangerous" since these are quite well captured on the mean field level. Although there is theoretical bluff involved, there are good indications that this "jellium" Fermi-liquid has quite some extra stability given the long range Coulomb interactions in the UV.

But it was realized perhaps as early as in the 1930's that this logic is severely impeded by the presence of a strong external potential coming from the ionic lattice. In chemistry this is known as the "Heitler-London" description of e.g. the H_2 molecule. One departs from two neutral hydrogen atoms a relatively large distance apart. The $1s$ atomic states do however overlap and these tunnel with a rate "t". When an electron tunnels one obtains a H^- atom and a proton. But now the big deal is that one can just measure how much energy it takes and one discovers that in H^- the two electrons are close together paying a Coulomb energy U which is actually very large, typically of order 10 eV. Different from the single particle description, the fluctuations where the electrons are exchanged are now associated with high energy virtual processes. The ground state becomes that of two protons both with an electron tied to it. But the electrons can only exchange virtually when their spins are antiparallel because of Pauli blocking and at low energy all that remains are localized spins on every hydrogen subjected to an antiferromagnet "superexchange" such that these form in the ground state a two spin spin singlet characterized by an exchange interaction $J = t^2/U$ as follows from second order perturbation theory controlled by the small "strong coupling" parameter t/U.

Notice that the Mott-insulator is really nothing else than a traffic jam occurring in the electron traffic. It is potential energy dominated and no funky quantum mechanical wave function effects are required as in band insulators which are rooted in quantum mechanical interference. Although worked under the rug for a long time by the (single particle) band structure community, this "large U" physics is actually quite ubiquitous in the zoo of solids. In essence, anything that is insulating having a nice colour is of this kind: the Mott insulators formed from transition metal and rare earth 'salts' (like $FeCl_2$, MnS, NiO, La_2CuO_4, Ce_2O_3

...). It turns out that for 3d- or 4f electrons the "effective U's" after incorporating the additional screening processes in solids can be typically still larger than the bandwidth.The minimal model capturing the essence of this "big number" physics is the celebrated fermion Hubbard model,

$$H = \sum_{\langle ij \rangle \sigma} t c^{\dagger}_{i\sigma} c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow} \tag{42}$$

This describes a lattice of tight binding electrons with a hopping t with a "H_2" repulsive interaction: when two electrons are on the same site they have to pay a Coulomb penalty U. Despite its simple appearance more papers seem to have been published dealing with it than even for the standard model of high energy physics!

There is now a general consensus that this is the guinea pig of the sign problem. This model became the focus of attention by the discovery of superconductivity in copper oxides in the late 1980's. It became clear soon after the discovery that these "cuprates" are *doped Mott insulators*, see Fig. (1). This departs from "parent" materials that are insulating, which are subsequently doped similar as to doping semiconductors.

The point of departure is thus the Mott insulator. The spins of the electrons are left behind at low energy and these are subjected to antiferromagnetic exchange interactions. Accordingly, an antiferromagnetic insulator is expected to form and this was indeed confirmed in no time in the parent cuprates. The parent Mott-insulators are in fact the only aspect of the cuprates that is thoroughly understood. Hell breaks loose for the theorists when these are doped. After countless attempts to get a mathematical handle on the doped systems, it gradually turned into the exercise ground for the "big machine" computational efforts aimed at breaking in the sign problem.

Before turning to the specifics of these efforts, so much is clear that the computers tell us that the sign problem in doped Mott insulators is particularly severe. It is not widely disseminated that in fact the origin of this trouble can be *diagnosed* although this does not help at all to find a cure. This is due to the Tsinghua theorist Zheng-Yu Weng who already started to realize the quantum statistical troubles in the mid 1990's with his "phase strings" – I find this so remarkably that I coined the name "Weng statistics" for it [35] .

This becomes transparant using once again the first quantized path integral language with its winding encoding of the quantum statistics. The root of the trouble is that in the Mott-insulator itself one is dealing with spins and spins are distinguishable particles with a Fock space consisting of tensor product states that are not anti-symmetrized. The reason is that spins are localized fermions which cannot exchange with each other – the virtual exchanges turn into the effective spin-spin interactions. However, when an electron is removed by doping the hole can move around and for this reason the electrons in the immediate vicinity start to remember that they are indistinguishable fermions that need anti-symmetrized wavefunctions. One is dealing with "part time" fermions that surely do not submit to the free winding affair of non interacting fermions!

One can actually find out what takes the place of the free winding. This is quite straightforward employing the high (physical) temperature expansion in the worldline language [36]. One departs from the $t-J$ model, the effective theory at energies $<< U$, describing holes hopping (with t) through the Heisenberg spins system (with super-exchange J) avoiding double occupancy in the process. For counting purposes one appoints the up spins as reference vacuum state, while the world histories of the down spins ("spinons") and holes ("holons") are tracked keeping the sign associated with a particular worldhistory separate.

One then derives for the structure of the quantum partition sum

$$Z_{t-J} = \sum_c \tau_c \mathcal{Z}[c] \tag{43}$$

Here $\mathcal{Z}[c]$ is the absolute value ("positive probability) of the contribution of a particular world history c formed from spinon- and holon worldlines. But the *sign* of this contribution is given by

$$\tau_c \equiv (-1)^{N_h^{\downarrow}[c] + N_h^h[c]} \tag{44}$$

$N_h^h[c]$ is business as usual: this counts the number of mutual windings (exchanges) of the holon wordlines, these behave as normal fermions relative to each other. But the surprise is in the quantity $N_h^{\downarrow}[c]$. When a hole hops to a site where a down spin ("spinon") is present the latter electron hops in the opposite direction, a "holon-spinon collision". For a given world-history $N_h^{\downarrow}[c]$ counts the total number of collisions. One adds these two up, and the parity of this integer than determines whether the worldhistory contributes with a "positive" or "negative" probability to the partition sum.

One discern immediately the difference with the sign structure of the free fermion problem.

There we found that the winding is hardwired making it possible to sum the alternating series into a Fermi-Dirac distribution. But in the doped Mott-insulator it becomes a *dynamical* affair. The negative probabilities are like destructive interference pushing up the ground state energy – the origin of E_F as the zero point motion energy of the Fermi-gas. But in the doped Mott-insulator one can organize worldhistories in such a way that the negative signs are avoided as much as possible. But this may in turn reduce $\mathcal{Z}[c]$ having also the effect of pushing up the energy. There is just no way that one can keep track of this balance with stochastic polynomial complexity (i.e. Metropolis) means.

This "minimization of negative signs" does to a degree depend on the way that the spin system is organized. It is easy to check that in an ordered, conventional antiferromagnet that can be stabilized by adding an external staggered magnetic field departing from a bipartite lattice *all* $N_h^{\downarrow}[c]$'s are even. The sign is then determined by $N_h^h[c]$. The outcome is that small "Fermi-pockets" are formed [35] with a Fermi surface spanning up a Luttinger volume proportional to p, the hole density. Departing from $U = 0$ one would find instead a volume $\sim 1 + p$ since one is dealing with a nearly half filled band. This is precisely coincident with what is found using straightforward Hartree-Fock mean field theory in the presence of antiferromagnetic order also for large U. Surely, this sign counting is completely general and therefore the hole pockets of conventional mean-field theory are just a special case.

But there are also quite different games one can play. For instance, one can contemplate that the spin system is in a resonating valence bond state ("RVB"), introduced early on by Phil Anderson merely on basis of his intuition. Thirty years down the line I am still waiting for either solid mathematical- or experimental evidences that it makes any sense. Whatsoever, the idea is to form singlet pairs of neighboring spins ("valence bonds") to then form a "maximal" coherent superposition of all possible tilings of the lattice by such valence bonds. The trouble is in the assertion that everything quantal in magnetism should be stitched together from the two spin "Bell pairs", the same kind of "two-ness" intuition that has been clouding the general understanding of many-body entanglement later in the quantum information community.

The original RVB idea was then that these valence bonds stay intact when the system is doped. In the presence of holes these turn into charged electron pairs that can delocalize and Bose condense, and this could explain superconductivity-at-a-high temperature. Yet again, no shred of evidence ever substantiated for this claim. But Zheng identified an interesting loop hole: it turns out that departing from such an RVB in the form of an Ansatz the number of negative signs is strongly reduced lowering the ground state energy. This suggests that it is perhaps such a bad idea, although one better be aware that the sign problem may be the culprit.

E. Epilogue: the big computational guns.

Computers keep people honest – for quite a long while the sign problem was worked under the rug by a large part of the physics community by framing it as a technical problem for software engineers. In fact, in the computational community there has been a consensus for a long time that this problem is fundamental.

In the stoquastic sector quantum Monte Carlo has been victorious. I already alluded to ^{4}He, we will touch on the role it played in the stoquastic criticality of the next section, while the computation of the difference in proton and neutron mass by "lattice QCD" can be considered as the ultimate benchmark. However, in a concerted effort of admirable perseverance that has been going for some 50 years computational physicists have been chasing the sign problem.

In recent years there has been noticeable progress. A first venue is straight QMC. The representation of choice is not the first quantized affair that I highlighted in the above. Instead, it turns out that the "determinant QMC" is much more efficient. This departs from the second quantized path integral – standard material in the advanced text books. One spans up the Hilbert space with generalized coherent states, mapping the second quantized canonical theory to the field-theory style path integral where the fermions are encoded by Grassmann (anticommuting) numbers. The fermion interactions are taken care of by the Hubbard-Stratonowich auxiliary fields such that the fermions can be integrated out. At the classical saddle point this turns into the standard Hartree-Fock (BCS etcetera) mean field theory, where the auxiliary fields are recognized as the order parameter. The effective action for these fields are however sign-ful through the fermion determinants. As the first quantized PI's, these are finite temperature methods defined on the Euclidean space-time cylinder.

The algorithms were developed already in the mid 1980's but with the computational resources available back then one could not get at temperatures well beyond the (bare) Fermi temperature before the exponential complexity brought it to a hold. But this has partially off set by the exponential growth of these resources – Moore's law. In the present era it is possible to get to much lower temperatures, that have started to overlap with the temperatures relevant for experiment, like 600 K or so dealing with cuprates [37]. Interestingly, the outcomes for the Hubbard model do not seem to connect that well to experiment: the case is developing that plain-vanilla Hubbard models may fall short to explain the physics of the strange metals.

The other progress has been in the arrival of algorithms that are inspired by quantum information, revolving around the many-body entanglement. These are the "tensor network" methods. This departs from the canonical formalism. For simplicity imagine a problem that is casted in terms of qubits, or equivalently $S = 1/2$ spins. The Hilbert space is spanned by "bit strings", tensor products of these local DOF's and one can write any energy eigenstate state as

$$|\Psi\rangle_k = \sum_{i_1,i_2,\cdots i_N} C_{i_1,i_2,\cdots i_N} |i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_n\rangle \tag{45}$$

where every local bit $|i_k\rangle$ can take two values. The amplitudes $C_{i_1,i_2,\cdots i_N}$ form a set of 2^N numbers – this is of course the quantum supremacy trouble. But these are evidently also the coefficients of a tensor C with N indices $i_1, i_2, \cdots i_N$ where each of them can take 2 values. It is a rank N index with 2^N coefficients.

The idea of tensor networks departs from the mathematical fact that the rank N tensor can be replaced by a "network" of smaller rank tensors. For instance, consider a chain of sites i. We can write in full generality C in terms of local rank 2 tensors (matrices) A as $C_{\cdots i-1,i,i+1 \cdots} = \sum_{k=1}^{2^N} \cdots A_{i-1}^{k_{i-2}k_{i-1}} A_i^{k_{i-1}k_i} A_{i+1}^{k_i k_{i+1}} \cdots$. In a spatially homogeneous system the A_i^k are the same, and all the data can be stored in this local tensor. However, the extra k label is referred to as the bond dimension: when one contracts the k's over the full 2^N Hilbert space the local tensors A contain all the information required to reconstruct the full tensor C. The rank of A is determined by the connectivity of the Hamiltonian – dealing with a e.g. a square lattice with nearest neighbour interactions a rank three tensor is required, etcetera.

The advantage is now that the *bond dimension encodes the "range" of the entanglement.* Take only $k = 1$ and you get a product state. When k takes two values you wire in nearest pairwise entanglement and so forth. Hence, these can be viewed as a particular variational

Ansatz that is constructed to regulate the entanglement. For small bond dimensions it describes SRE product states with only very local entanglement and by gradually increasing the bond dimensions one wires in an increasing degree of many body entanglement and one tracks how the outcomes are evolving. Of course, there is just ”conservation of misery” dealing with quantum supreme states.

In combination with an efficient algorithm to optimize the Ansatz the matrix version suited for one space dimension of the above is the ”Density Matrix Renormalization Group” (DMRG) introduced by Steve White in the early 1990’s [38] when it was not even quite realized that it is controlled by the degree of many body entanglement. Although limited to systems having eventually a 1D connectivity this turned into the intervening period in a success story (e.g., [39]). In addition, a substantial portfolio of other tensor networks was constructed, designed to deal with specific circumstances that are harder to handle for all kind of computational reasons (e.g., ”IPEPS” for two dimensions, ”MERA” for scale invariant problems).

QMC and the tensor networks do have the benefit that they reveal their range of applicability. In addition, there are more ad-hoc methods that do invoke uncontrollable assumptions. A first category are implementations of Ceperley ”fixed nodal surface” as discussed in the above. Another important category is resting on the ”dynamical mean field theory” (DMFT) idea. Dealing with short ranged models (like Hubbard) one can show that in *infinite* dimensions the problem reduces to a single strongly interacting (”Kondo”) impurity in an effective medium – although quite untrivial such impurity problems can be solved using e.g. Quantum Monte Carlo. There is literally nothing lying on the shelf telling that these solutions have anything to do with the physics in 2 or 3 dimensions. However, one can just implement a cluster of increasing size feeling the effective medium boundary conditions and see how it evolves: the ”cluster” DMRG.

What is the state of the art? This is encapsulated by a recent large scale effort (the ”Simons collaboration”) where the whole repertoire of different methods as discussed in the above was unleashed on the Hubbard model for large U and a doping around 1/8 [40]. Although quite different systematic errors are involved, eventually all these models arrived at the same answer: the ”spin stripes” of the kind that I discovered in 1987 using plain-vanilla mean field [41]. It is a bit of a sideline in the big picture, and I discuss these in a bit more detail in Section (VIII).

Such "stripes" are insulating and SRE products – this just reveals that there are strong ordering tendencies in the Hubbard model. But such stripy states are just a small part of the whole story. Presently it appears that the most controlled results are produced by DMRG for Hubbard type Hamiltonians defined on a ladder geometry: the legs are associated with one dimensions where DMRG works very well. One can then gradually increase the width (putting "rungs" between the legs) paying the prize that the computational complexity grows exponentially in the width. Even for a width 4 it turns out that one has to invoke a very large bond dimension (like $\sim 10^4$) to get convergence. But yet again, it does not shed light at all on the strange metal UV [39]. One finds at long distances that Luttinger liquid universality takes over, an affair which is completely tied to one space dimension. Quantum statistics is meaningless in 1+1D and the ramification is that one can always identify a "sign free" representation. Subsequently, in one dimension everything scales to strong coupling and algebraic long range order takes over: the secret of one dimensional physics as enumerated in the 1970's.

V. WHAT YOU ABSOLUTELY NEED TO KNOW ABOUT THE ADS/CFT CORRESPONDENCE.

We have arrived at the last collection of preliminaries as required to appreciate the substance of these lecture notes. This is not intended to be a primer on the AdS/CFT correspondence. This is a mathematical contraption that rests on 40 years or so hard work by string theorists, being so remarkable that a large part of the contemporary activity in this community revolves around the "correspondence". In the 20+ years after the discovery by Maldacena in 1997 thousands of papers have been published turning it into a vast subject. In the particular corner of interest here – the Anti-de-Sitter to Condensed Matter Theory, "AdS/CMT" – three bulky textbooks are available which are complementary in their focus [8–10].

To learn how this works may take a year of your life. Stronger, to operate at the frontier of the development you better engage in it for your PhD thesis in order to build up a big repertoire of all the technical tricks of the trade. I learned it an elevated age: I do believe that I have a fair understanding of how it all works but I am heavily dependent on skilled coworkers to get the computations done.

I will explain in one concise section what you really need to know in order to appreciate the power it exerts on thinking out of the established condensed matter box that I exhibited in the above. Handle it in the same way as a theorist deals with experiment. You don't want to know how the op-amps and vacuum pumps of the laboratory rigs precisely work, but you do need to know what kind of specific information the experimental colleagues can deliver, and especially the caveats and restrictions that are invariably met in the real world.

A. The allure: fancy black holes as quantum computers.

AdS/CFT was not meant to shed light on electron systems. All along the development of string theory was propelled by the promise that it could shed light on quantum gravity. It is really an outgrowth of quantum field theory, dealing with extended objects (strings, branes) that are infused with great amounts of symmetry (supersymmetry, Weyl-invariance, $\cdots$). In a mathematical miracle, results dropped out hinting at surprising relations with "gravity", Einstein's theory of general relativity (GR). This really landed on its feet with Maldacena's discovery.

We are witnessing presently the second youth of GR. It started with the cosmology revolution in the 1990's, where this somewhat flaky affair turned into a quantitative science due to the high resolution mapping of the CMB and so forth. More recently the gravitational wave detectors got on line and the astronomers pulled off the observation of the supermassive black hole shadow with the event horizon telescope. This is testimony of the mathematical quality of GR: the math forced upon us the wisdom that black holes are like the "atoms" of space time and half a century later the hardware was developed to a degree that it could be confirmed by observations. Black holes are presently at the centre of attention.

You have to be quite fluent both in GR and quantum theory to recognize the allure that is eventually responsible for the prominence of the correspondence in the string theory community. The equations that tell you that black holes exist are completely different from anything in the quantum theory, not to speak about the phenomena they describe. It is then a revelation to find out that the black hole equations can be used to reconstruct anything that really matters in the established condensed matter agenda. But it reaches much farther.

My claim is that *state of the art black hole mathematics acts as a quantum computer revealing general phenomenological principles governing observable properties of quantum* *supreme matter.* In fact, the ultimate promise is that by establishing this connection more thoroughly – it is still under construction – there is a potential that experimental work will reveal surprises that may impact in the "reverse gear" of holography: shed light on the deep mystery of quantum gravity. Frankly, compared to this challenge explaining why the superconducting Tc is high (the holy grail in traditional condensed matter) is worthy no more than a footnote.

A big part of the technical hardship I referred to in the above is due to the fact that the gravitational physics that is "dual" to the quantum matter is state of the art. The black holes at the centre of public attention originate in the 1960's – the Kerr solution. These are quite simple for the reason that it was well into the 21-th century taken for granted that "black holes cannot have hair". In an asymptotically flat space time stationary black holes have to be featureless according to the no-hair theorem: a black hole is like an elementary particle that can only be characterized by its overall energy, (angular) momentum and electromagnetic charge. Admittedly, dealing with dynamical circumstances such as black hole mergers the Einstein equations representing a system of non-linear partial differential equations (PDE's) come to live. It took actually half a century or so to tame these horrible equations to a degree that supercomputers can handle them – the numerical GR that plays a crucial role in interpreting the GW detector signals.

Spurred actually by the AdS/CMT agenda, it was a decade ago realized that under a general condition tied to the correspondence – the gravitational space time has to be asymptotically Anti-de-Sitter – there is no such thing as a no-hair theorem. In a flurry that followed the string theorist pulled off a hitherto unexplored new area in GR. Equilibrium in the quantum theory corresponds with stationary gravitational solutions, and the holographists discovered a zoo of fanciful black hole hairs. The state of the art is that one needs the numerical GR technology running on supercomputers to find out how such "rasta hair" black holes look like. This present frontier will be highlighted in particular in Section (VIII).

B. The plain vanilla AdS/CFT correspondence.

To get an idea of how the correspondence works, let us start focussing in on the bare bones version: the "set up" discovered by Maldacena. This was in fact a spin-off of the so-called "second string revolution". This is a remarkable story involving a series of profound mathematical discoveries that is beyond the scope of these notes. What you need to know is that this correspondence is regarded as a mathematical fact, proven at least to physicists standards. This theorem is as follows: "Maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills in D = 4 space-time dimensions in the large N limit for infinite 't Hooft coupling is *dual* to classical supergravity on $AdS_5 \times S^5$." What has this string-speak to do with electrons in solids?

Let's start out with the $\mathcal{N} = 4$ Yang-Mills part. Quantum chromo-dynamics with its quarks and gluons is an example of a Yang-Mills theory. The quarks carry three colour charges ($N = 3$) and there a total of $3^2 - 1 = 8$ gauge bosons (gluons) exchanging the colours between the quarks. The mapping to *classical* (computable) gravity claimed by the correspondence requires that this number of colors is sent to infinity. The crucial part is that one is dealing with a *matrix* field theory (the N^2 gluons) and this large N limit is a crucial condition.

Supersymmetry on the other hand is not crucial: a zoo of non-supersymmetric correspondences were identified later. But there is one particular feature that is important. Maximal supersymmetry means that the fermionic- and bosonic fields are in perfect balance at zero (quark) density and this leads to the "non-renormalization theorems": departing from a scale invariant free theory the fermion- and boson contributions in the RG equations cancel each other exactly and this has the implication that one no longer has to fine tune to a phase transition to obtain a perfect conformal invariance. This just describes a quantum critical state as discussed in the previous section. The "large 't Hooft coupling" has now the meaning of "strongly interacting critical" where infinite 't Hooft coupling means as "strongly interacting critical as possible".

Such theories are giving in to the rules explained in the previous section: two-point propagators are branch cuts characterized by anomalous dimensions and a-priori these can be very anomalous. But the large N CFT's are actually not at all understood in the canonical language of quantum information. There are however good reasons to believe that the combination of large N and infinite 't Hooft coupling implies that the many-body entanglement is pushed to its very limit. This is the important reason to pay attention: the string theory speak is just coding for a maximally "quantum supreme" stoquastic state of matter – as always in physics one wants to know first the limiting cases and this Yang-Mills affair is about the dense entanglement limit.

The crucial word in the definition of the correspondence is ”dual”. Metaphorically this is similar to the quantum-mechanical particle-wave duality. ”Particle” is in a way opposite to ”wave” but we understand too well that these are two sides of the same coin. There is the quantum-mechanical ”wholeness” and pending the way one observes the system one percieves it as a particle- or wave physics. In fact, AdS-CFT is more closely related to the field-theoretical weak-strong (”S”) duality: the Kramers-Wannier type of affair where the ”strongly coupled” (disordered) state corresponds with an ordered state of the disorder operators (topological excitations) of the weakly coupled (ordered) state. The Yang-Mills theory is like the strongly coupled affair, and the gravity side is like the ”order by disorder operators” that we encountered in the previous sections. But there is one extra feature that is beyond anything that is found in the field-theoretical dualities. It is a *holographic* duality.

The word refers to the familiar ”holograms”: one has a two dimensional photographic plate with complicated interference patterns, and upon shining through a laser beam it reconstructs into a recognizable three dimensional image. The big deal is that all the information for the 3D image is encoded in one lower dimension, be it in the form of completely incomprehensible interference fringes. In this comparison, the Yang-Mills theory is like the photographic plate and gravity is the three dimensional figure. It is in more than one regard an effective metaphor. Somehow, the gravitational side is after some training becoming intuitively comprehensible for our ape-brain: black holes are employed with great effect in Hollywood movies. But the quantum world is inherently abstract, like the interference fringes.

The real depth is however in the ”holographic principle” formulated in the early 1990’s by ’t Hooft and Susskind stating that the number of degrees of freedom in a gravitational theory relates to that in a quantum field theory as if the former lives in a space-time with one less dimension compared to the latter. This is rooted in semi-classical black-hole wisdoms: the famous black hole radiation by Hawking. Depart from a classical (Schwarzschild) black-hole space-time and just insert quantum fields, to discover that the black hole turns into a black body radiator according to external observers. Similarly one can associate a Bekenstein-Hawking entropy to the black hole and this scales with the *area* of the black hole horizon. Field theoretical objects have an entropy scaling with the *volume*: here is the ’missing dimension’.

Part of the 1997 excitement was in the fact that AdS/CFT was the first mathematically

$$dr^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad F(r) = -\Lambda r^2 + 1, \qquad \Lambda < 0$$

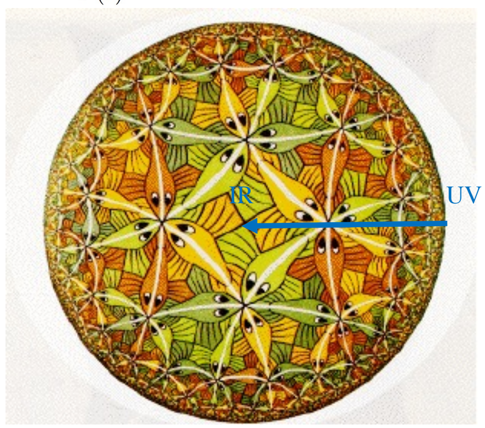

FIG. 9. Esher's graphical representation of the Poincare disc, a two dimensional cut through the hyperbolic Anti-de-Sitter space. The radial direction (blue arrow) is the extra dimension encoding for the scaling direction in the quantum field theory that lives on the boundary of AdS.

controlled ploy that confirmed the holographic principle. The field theory lives on the 3+1 dimensional "boundary" while the gravitational dual lives in the "bulk" 4+1 dimensions: the "5' in AdS_5 while S^5 refers to 5 extra dimensions that are rolled up (compactified) in tiny little circles.

But this gravitational space-time is of a special kind: it has an "Anti-de-Sitter" geometry. This is a maximally symmetric manifold characterized by a constant *negative* scalar curvature. It is a solution of the vacuum Einstein equations for a negative cosmological constant. It is a hyperbolic space with a curvature that can be represented by the "Poincare disc" representing the extra "radial dimension" (relative to the boundary) and in addition one of the space dimensions that its shares with the field theory. This was in turn famously visualized by the artist Esher with the "fishes", Fig. (9). The radial direction is the radial coordinate of this disc (blue arrow). Strangely, the drawing gives away that this is an infinitely large space having still a boundary: appealingly, the field theory lives on this boundary.

Rooted in the Lorentzian signature one also encounters a typical "causal weirdness" motive which is actually crucial for the correspondence to work. Although the AdS boundary lies infinitely far away from the deep interior as measured along the radial dimension, a signal emitted in the deep interior arrives at the boundary after a *finite time lapse*. This is actually a necessary condition for the correspondence to work: the long wavelength physics in the boundary is determined by the bulk geometry in the deep interior and a causal connection is required to make this work.

The figure also reveals a sense of fractality, scale invariance: the pattern of fishes repeats itself but the fishes shrink in size going from the middle (”deep interior”) to the boundary. But this is not symmetry in the usual sense: it is ”regularity” associated with the curvature of the manifold which is called ”isometry” by the geometers. One meets here a deep and rather recent mathematical insight: the *isometry* of a curved space in d+1 dimension is linked to the *symmetry* encountered by e.g. field theories in d dimensions. This AdS ”fractality” of Esher's drawing is encoding for the conformal invariance of the boundary field theory. Therefore, ”Anti-de-Sitter (AdS) corresponds with Conformal Field Theory (CFT)”.

The final defining characteristic is the ”dictionary”, prescribing in a precise mathematical fashion how to relate quantities in the boundary to those in the bulk, metaphorically like the Fourier transformation formula behind the particle-wave duality. This is derived from the ”G(ubser)-K(lebanov)-P(olyakov)-W(itten)” rule that was discovered soon after Maldacena's demonstration. This can be written in a concise form as,

$$\langle e^{\int d^{d+1}x J(x)O(x)} \rangle_{CFT} = \int \mathcal{D}\phi e^{iS_{\text{bulk}}(\phi(x,r))|_{\phi(x,r=\infty)=J(x)}} \tag{46}$$

The left hand side refers to the boundary. This is the generating functional yielding the information to compute propagators like $\langle OO \rangle$ by taking functional derivatives $\delta/\delta O(x)$. The right hand side represents the full quantum partition sum of the dual *quantum* gravity problem in the bulk. But the specialty is that the bulk field ϕ that is dual to O in the boundary is constrained at the boundary of AdS to be coincident with the source J of the boundary theory. Upon taking the large N and 't Hooft coupling limit the r.h.s. turns into the classical saddle point – this is just the usual business of deriving the Einstein equations by extremizing S.

A minimal example of how this works can be looked up in the introduction of Ref. [9] and let me sketch here the outcomes. Depart from a scalar field ϕ with mass m in the bulk and solve its EOM's in an AdS background. You will find that upon approaching the AdS boundary (radial coordinate $r \to \infty$) this field falls off universally according to $f_k(r) = r^{-d-1+\Delta} A(k) + r^{-\Delta} B(k)$ where k refers to the energy-momentum in the boundary while $A(k)$ and $B(k)$ are the coefficients of the leading- and subleading components of ϕ.

As it turns out, the GPKW rule implies that the two point propagator of the operator O in the boundary is given by $\langle O(-k)O(k)\rangle = B(k)/A(k)$, the ratio of the subleading $B(k)$ (”response”) to leading $A(k)$ (”source”) boundary asymptotes of ϕ in the bulk. It follows that $\langle O(-k)O(k)\rangle \sim k^{2\Delta-d-1}$. This is just a branch-cut, as you learned to appreciate in Section (III B). Δ is here the scaling dimension of the operator O, according to the conventions of conformal field theory. This is determined by the classical EOM governing ϕ: $\Delta = \frac{d+1}{2} + \sqrt{\frac{d+1)^2}{4} + m^2L^2}$ where L is the AdS radius (setting the overall scale) and m the mass of the bulk field. The take home message is that the *mass* of the field in the bulk is dual to the *anomalous dimension* of the operator O in the boundary! It is in fact bounded by $m^2 = 0$ is zero where one finds the ”engineering” dimension of the free critical theory, Eq. (18). Such massless fields are typically associated with conserved currents in the boundary (like total momentum in the homogeneous space).

This illustrates the rather counterintuitive ”magical mechanism” implied by the GKPW rule. Upon further exploring it, one finds that pure AdS encodes for all the generic traits of a Lorentz-invariant, zero density strongly interacting quantum critical state of the kind as you met in Section (III B). However, the structure of the duality permits the anomalous dimensions to become anything: it is just pending the mass of the bulk field. These are fixed departing from ”top-down” precision set ups. But what matters here is that the zero density point of departure – the pure CFT – is of the strongly interacting ”quantum-supreme” kind as testified by the anomalous dimensions.

The *structure* of the renormalization group flow with its ensuing ramifications for the boundary (the scaling theory) is hard wired in the geometry of the bulk and only the numbers (anomalous dimensions) are pending the specifics of the UV theory (large N, etcetera). The take home message of the remainder will be that this pertains to the much more intricate circumstances one meets at finite temperature and especially the ”signs infested” finite density circumstances. The above example is the most elementary GR exercise one meets in the holographic duality agenda. Upon rolling out the amazingly rich full AdS/CMT portfolio the basic moral of this example does however repeat itself systematically. Some intricate property of the boundary theory under ”real life” conditions translates typically into a gravitational problem to solve in the bulk that you would not guess at all beforehand. In no time one runs into fancy, state of the art GR computation and I will keep this largely implicit in the remainder, at best I will sketch it in words. To get into it seriously, study the books and do the exercises.

However, there is one other dictionary entry that is rather easy to understand conceptually and quite entertaining: how to deal with finite temperature? The holographic magic that I just pointed out turns out to work amazingly well. Let me conclude this very short introduction in AdS/CFT with a sketch how this works.

C. Plain vanilla AdS/CFT and finite temperature.

It was early in the development realized by Witten how to encode the boundary finite temperature in the bulk. The answer: insert a Schwarzschild black hole in the deep interior (the "middle") of AdS! This excited the string theorist because it linked AdS/CFT directly to the portfolio of black hole quantum physics: Hawking radiation, Bekenstein-Hawking black hole entropy and so forth.

I already alluded to the holographic principle inspired by the Bekenstein-Hawking entropy of the "thermal" black hole. This set of ideas is elementary and let me summarize the essence in a bit more detail. Take a classical, Schwarschild black hole geometry and implement free quantum fields in this space time. Through the combination of a causal horizon and an "accelerating" geometry near the horizon (Unruh effect), using the standard rules of quantum mechanics an external observer will perceive the black hole as a black body radiator with a temperature set by the *area* of the horizon. One can deduce the entropy of such a thermal object and this is the Bekenstein-Hawking entropy. One divides the horizon in cells with a dimension set by the Planck length, inserting one bit of information in each Planck-cell: the entropy of a macroscopic black hole is actually very large. But what matters most is that this entropy with the *area* of the horizon and not with the *volume* that the black hole occupies. In quantum field theories it always scales with volume and it is as if a dimension is missing in the gravitational theory. This is the root of the idea behind the "holographic" principle: 't Hooft conjectured that this a generic trait of quantum gravity theories, and the idea was further elaborated by Susskind.

This holographic entropy affair gets bigger than life in the correspondence. Mathematically this is a surprisingly elegant affair. Insert a Schwarschild black hole and Wick rotate both in the boundary and the bulk to Euclidean signature – the bulk black hole is stationary encoding for equilibrium in the boundary. GR is a-priori not pending the signature of time –it is just an independent input. One can evaluate the Schwarschild metric in Euclidean signature by analytical continuation and the outcome is as follows. Focussing on the radial direction - imaginary time coordinates one finds that the latter rolls up in a circle that shrinks to zero radius at the horizon. This circle opens up moving away from the horizon to acquire a radius set by the Hawking temperature at infinity. But the AdS boundary is at infinity: the boundary field theory lives in a geometry where the imaginary time forms a circle with radius $\tau_\hbar = \hbar / k_B T$. This of course rings a bell, Section (III C 2)! Yet again the "isometry = symmetry" logic is at work: the "finite size scaling" effect of temperature in the boundary is encoded entirely in the geometry of the bulk black hole.

The bottom line is that all known principles of the boundary thermodynamics are impeccably reproduced by the bulk thermal black holes. The entropy counts in the correct way in the elementary CFT setting as it should, according to $S \sim T^d$. But we will find out that dealing with the rich "hairy" black holes of AdS/CMT the thermodynamic principles associated with phase transitions are impeccably reconstructed as well. E.g. the thermodynamics of holographic superconductors near the thermal phase transition is precisely captured by the Ginzburg-Landau free energy functional (Section VI E 3).

Perhaps the most impressive result is that this success also extends to the dissipative *dynamics* of the macroscopic fluid formed at finite temperature. For any finite temperature system at macroscopic times living in a homogeneous and isotropic background we have a universal theory that dates back to the 19-th century: Navier-Stokes hydrodynamics. This describes the responses of the fluid under non-equilibrium conditions. The Lorentzian signature now matters: one has to evaluate gradient-expansion "space-time quakes" in the near horizon geometry encoding for the "deep IR" (macroscopic times and length) in the boundary. Upon pulling this via the dictionary to the boundary the outcome is – it impeccably reproduces the mathematical *structure* of Navier-Stokes! Hydro is therefore a special corner of GR. This actually poses a significant problem to the mathematicians. There is a Clay institute "milennium prize" on Navier-Stokes but not on GR ...

However, the parameters of the hydrodynamical theory are not standard departing from AdS/Schw geometry (Schwarzschild Black-hole in AdS). The crucial part is the dissipation, encapsulated by the *viscosity*. Departing from a conformally invariant UV the bulk viscosity has to vanish for symmetry reasons (be aware of such "UV dependence") and one is only dealing with *shear* viscosity. This can be easily computed in linear response and the outcome is yet another simple marvel. Shear viscosity is sourced by shear stress and the dictionary insists that this is encoded by *gravitational waves* ("gravitons") propagating in the bulk. Anything dissipative in the boundary is dual to things that fall through the horizon in the bulk. The viscosity is therefore dual to the *absorption cross section of zero frequency gravitons by the black hole*. This is obviously proportional to the *area* of the horizon. But the entropy density s is also proportional to the area and the bottomline is that they are related by a geometric $1/(4\pi)$ factor. The result is the "minimal viscosity",

$$\frac{\eta}{s} = \frac{1}{4\pi}\frac{\hbar}{k_B} \tag{47}$$

Can it be simpler? Viscosity is just set by $\hbar$ and the entropy. As it turns out, this ratio is extremely small as compared to what is established in typical normal, "molecular" fluids like water and so forth. But the crux is of course that this is a classical fluid departing from the strongly interacting quantum critical state and we learned about the intrinsic extremely rapid thermalization in Section (III C 2). In fact, the minimal viscosity is just Planckian dissipation in disguise.

This follows from elementary dimensional analysis. The dimension of viscosity is set by the free energy density $f(T)$ and the characteristic momentum relaxation time associated with the presence of finite spatial gradients, τ_P: $\eta(T) = f(T)\tau_P$. Dealing with a quantum critical state there is no internal energy and the free energy is entirely entropic $f(T) = sT$. Stick in $\tau_P \simeq \hbar/(k_B T)$ and it follows that $\eta \simeq (\hbar/k_B)s$! One needs the black holes (and large N) only to determine the parametric factor $1/(4\pi)$.

Notice an aspect that may be at first sight confusing. Planckian dissipation means "the best heater that can be realized in principle". But this leads to an extremely small viscosity, which is in turn governing how the moving fluid converts its kinetic energy into heat. The secret is in the fact that viscosity is associated with momentum dissipation and it is proportional to the very short Planckian time and not the Planckian rate. For instance, the viscosity in the ^{3}He becomes very large in the low temperature Fermi-liquid regime, a main challenge facing the design of dilution fridges. The reason is that the viscosity in the Fermi-liquid is set by the quasiparticles collision time τ_c, the scale associated with the exchange of momentum. This scales like $\tau_c \sim E_F\hbar/(k_B T)^2$, becoming very long at low temperature because of the extra E_F/T factor compared to $\tau_\hbar$.

The minimal viscosity as realized in 2002 forms the starting point of what later turned into the AdS/CMT portfolio. It caused a big splash in 2005 when evidences appeared that it is actually at work in nature. The context is the quark gluon plasma. This refers to an affair that has been heroically pursued for a long time in high energy physics. Collide heavy nuclei at a very high speed in high energy accelerators and the ensuing fireball may get so hot and dense that one gets in the deconfining regime of QCD. One anticipates that a plasma may be realized of free-ish quarks and gluons and when the explosion time permits this may eventually behave as a hydrodynamical entity.

Around 2005 firm evidences started to appear at the RIHC facility in Brookhaven that such a plasma is indeed realized, behaving in hydrodynamical ways. Remarkably, the experimentalists managed to measure the η/s ratio. Based on QCD perturbative theory (morally, kinetic gas theory) one anticipated a value for this ratio that is order of magnitudes larger than what was measured. Instead, it turned out to be very close to the minimal viscosity prediction of holography. It is still a bit shady why QCD in this regime should give in to "conformal principle" – QCD is about running- and not marginal couplings but one may envisage that at the temperatures reached in the heavy ion collision both the "asymptotic freedom" (kinetic gas physics) and "infrared slavery" (confinement) are "balancing" in an effectively "quantum supreme" physics.

Since 2005 this "fluid-gravity correspondence" turned into a subfield of its own. Among others it was used to sort out higher gradient hydrodynamics. Proceeding phenomenologically this turns into a complicated affair but using the gravity dual it becomes rather mechanical, solving the gravity equations systematically. In this way, it was used to detect for instance subtle flaws in the derivations of the Landau hydrodynamics school, but also shedding light on tricky themes like how to handle entropy currents in relativistic regimes. It also works the other way around. The small viscosity implies that turbulence is around the corner: this is dual to "turbulent" near horizon geometry in the bulk. This is technically quite hard to deal with but there are reasons for such turbulent horizon to be perhaps of relevance to black hole mergers.

All of this is described in detail in the textbooks [8–10]. The greatest intrigue is perhaps associated with very recent developments [7] that unfolded after the books were written. Using special "out of time" correlation function one can lay its hand on the short time "quantum chaos". One can identify a "quantum Lyaponov ("scrambling) time" that in essence how long it takes for an observable to get completely lost in the exponentially large Hilbert space. It can be shown that this is also set by $\tau_\hbar$ in the holographic fluids. Amazingly, together with the "butterfly velocity" expressing how fast this chaos spreads in space a holographic mathematical relation ("pole skipping") shows that the macroscopic viscosity is completely set by these short time chaos quantities! This is perhaps best understood as reflecting the very rapid eigenstate thermalization in these maximally many body entangled CFT's. In essence, the hydro that requires local equilibrium already sets in at the short scales of the chaos and nothing new happens until one reaches the macroscopic scale. It is actually a bit ironic that the belief is relatively widespread in the community working on fluid gravity that this may tell stories about hydro in general. I can assure that nothing of the kind is going on in water – it is all about quantum critical systems and the hydro associated with pure CFT's has not been identified in the laboratories.

D. Epilogue: the UV independence of the structure of the zero density scaling theory.

To conclude this section, the take home message is that the success of this thermal agenda raises the confidence in the "holographic oracle". Despite the large N etcetera special conditions formally required to rely on classical gravity, the correspondence automatically encodes for the correct *structure* of the phenomenological theories associated with the boundary. The *numbers* are in hindsight also reasonable – only parametric factors like the $1/(4\pi)$ in the minimal viscosity are pending manifestly on the infinite N limit.

I never got a clear answer why classical bulk gravity works so well in reproducing the universal aspects of this zero density affair. Anything that is going on in the physical universe is far from this large N limit, with its classical gravity bulk. As I emphasized, for "physical" small values of N the bulk is supposed to be governed by a quantum geometry that is still in the dark, but this does not seem to matter for the *structure* of the theory predicting physical observables – the branch cut propagators at zero temperature, the Planckian dissipation at finite temperature, the thermodynamics and so forth.

So much is clear that it is all controlled by *symmetry*, and zero density holography is just an impeccable symmetry processing apparatus somehow revealing the most general way this controls observables. The conformal invariance constrains the form of the thermodynamics, while it imposes the branch cut form on the $T = 0$ propagators leaving the anomalous dimensions as free phenomenological parameters. Even less obviously, it works as well for the finite temperature physics as implied by the finite size scaling of the Euclidean time circle, eventually turning into Navier-Stokes due to impeccable encoding of conserved Noether currents and the dissipation mechanism.

We will next dig into the fermion sign invested finite density holographic portfolio. In this area we cannot check it against facts we know directly from the QFT side since nothing is known for sure. Is the same magic at work as at zero density, perhaps indicating that in densely entangled quantum supreme matter symmetry principle is governing everything that can be measured to an even greater degree that we are used to in conventional semiclassics? As we will see, the nature of this "holographic" symmetry principle does change drastically at finite density translating in a different type of "quantum critical" phenomenology. But is it reliable? This is the big question that is begging for an answer.

VI. THE REVELATION: THE HOLOGRAPHIC STRANGE METALS.

We are done with the preliminaries. As announced in the introduction, at stake is whether universal emergent physical principle is at work dealing with quantum supreme finite density matter which lies hidden behind the fermion sign problem brick wall. Once again, this cannot be emphasized enough: because of the fermion signs we used to be mathematically blind.

However, based on general reasoning we can at least deduce some general conditions of such matter, based on avoiding no-go theorems. In the first place, such matter has to be *compressible*. "Incompressible" refers to the circumstance that the ground state is separated from all excitations by a finite, absolute energy gap. Much of the effort using tensor networks and related technology has been focussed on such gapped systems because there is much more control: the exponential complexity associated with the delocalization in the vast many body Hilbert space is inhibited by the presence of the gap. All that can survive is a very sparse form of many body entanglement that is captured by *topological field theory* [42]. Examples are the fractional quantum Hall physics captured by Chern-Simons theory and the deconfining states of discrete gauge theory underlying e.g. Kitaev's toric code.

Although not a closed chapter, it is fairly well understood: in essence, the sparse many body entanglement of the microscopic degrees of freedom translate into the "quantum-weird" global properties of the system such as the ground state degeneracy pending the topology of the space, the edge modes and so forth. One can identify "topological quasiparticles" such as the (practical) Majorana zero modes that satisfy non-abelian braiding properties so that they can be used as topologically protected qubits. But the big deal revealing the sparseness of this entanglement is that an *infinity* of microscopic "qubits" are required to form a single topological bit. Once again, given that the required math (topological field theory) appears to be available it is reasonably well understood and I refer to the extensive literature.

Hence, the presence of scale is bad for the "quantum supreme" entanglement and we may find inspiration from the stoquastic side. There we found that only when scale invariance is generated in the IR, at the critical point, the vacuum state delocalizes in an extensive part of the vast many-body Hilbert space. Is such an emergent scale invariance a necessary condition for finite density fermion matter to become quantum supreme? The central result of the holographic description of finite density "signful" systems is that this should be the case, while it delivers a "covariant' scaling theory that is of an entirely different nature than what you just encountered for the "stoquastic" CFT's. Different from the latter, this cannot be checked since the "sign brick wall" prohibits this to be evaluated in the boundary language. Hence, there is no mathematical theorem insisting that this has to be the case. But it is the central hypothesis underlying this whole venture, and the challenge is to test it using the presently available means: the condensed matter laboratories (Section IX).

A. How to reconcile scaling and the fermionic degeneracy scale?

The ingredient that appears to be unavoidable is the notion of a *fermionic degeneracy scale*. In the discussion of the Fermi-liquid in Section (IV) it may already have occurred to the reader that fermion signs may have the universal consequence that the ground state has to be characterized by a large zero point motion energy. In the case of the Fermi-liquid this is of course obvious: given the simple Pauli principle for free fermions it follows immediately that the Fermi-energy represents this zero-point motion energy. The first quantized "winding number" path integral offered a complimentary view. We found that the "negative probabilities" have the effect that as compared to the stoquastic case the ground state energy is pushed upward – the alternating-in-signs sum over winding numbers encoding for the Fermi energy.

Yet again there is no mathematical theorem but it seems impossible to avoid the wisdom that the signs will invariably raise the zero-point energy: fermionic quantum supreme states of matter should be characterized by a fermionic degeneracy scale. But we directly infer a tension: I just argued that quantum supremacy appears to require scale invariance but in a non-stoquastic problem one has to accommodate a degeneracy scale at the same time. At first sight these two requirements seem to mutually exclude each other.

But there is a loop hole. As I emphasized in Section (IV B 2), *all physical observables of the Fermi-liquid of the Fermi-liquid are power laws,* at least in so far these are not hydrodynamically protected. As becomes explicit in the functional RG description [34] , this is rooted in a continuous RG flow towards the infrared which is not interrupted by scales (gaps). But this flow is distinct from the RG encountered in the CFT's which is rooted in the *invariance* under conformal transformations. This is most notable in the fact that the Fermi-energy can be discerned from macroscopic measurements. E.g. the Sommerfeld specific heat tells immediately the value of the (renormalized) Fermi energy – in a CFT the specific heat only reveals IR data.

The take home message is that the holographic strange metals *share* with the Fermi-liquid this sensitivity to the fermionic "degeneracy" scale designated by the chemical potential μ. At the same time, all observables are power laws as well. However, the structure of the scaling theory is entirely different but this can already be discerned by inspecting the Fermi-liquid theory. In fact, this particular form of scaling exhibited by the Fermi-liquid was recognized *after* the holographic results became available.

The "(boundary) symmetry to (bulk) isometry" mathematical relation reveals at this instance its full powers. Perhaps I did not say it loud enough yet: the bulk geometry encodes for the RG flows, the renormalization group is geometrized in the bulk. This is concisely formulated as "RG = GR". Dealing with the finite density systems one finds an *emergent* geometry in the deep interior associated with the long wavelength physics in the boundary. The beautiful governing principle revealed by holography is that generically in the finite density systems the deep interior isometry is of a different fundamental nature than what is encountered with the CFT's. For the latter the metric is *invariant* under scale transformations imposing the correct scaling behaviour in the boundary that I just discussed. But for finite density, the metric is only *covariant* under scale transformation in the gravitational sense of the word: after a scale transformation the metric is *proportional,* *but not identical to itself.*

This principle rules the Fermi-liquid but also the holographic strange metals. The relation between them is at least metaphorically like the one encountered in critical theory. The Fermi liquid is like the free Landau mean field critical state encountered above the upper critical dimension. As governed by the "covariance" principle, the holographic strange metals are obeying a scaling theory structured in the same way as for the Fermi-liquid. The difference is that in the strange metals the "engineering dimensions" of the Fermi liquid turn anomalous!

Dealing with stoquastic criticality the scaling exponents associated with different quantities are organized in terms of a small number of underlying "primary" dimensions set by symmetry and dimensionality: the Kadanoff scaling relations. According to holography, the scaling properties of the strange metals are also "organized" but in a way that is entirely different from the stoquastic case. Resting on gravitational universality it appears possible to formulate universal scaling relations involving underlying dimensions of a completely different kind than the stoquastic ones. Macroscopic properties are claimed to be governed by the so-called *dynamical critical exponent* z, the *hyperscaling violation dimension* θ and the *charge exponent* ζ. The first one has an existence also in the stoquastic case taking however values that are unheard off in the statistical physics portfolio. The last two are unique to finite density.

Last but not least, the Planckian dissipation continues to rule the finite temperature responses, highlighted by the so-called minimal viscosity. Let me now take you step by step in this intriguing affair.

The climax of the story is in Section (VI D 3) where the "covariant RG" will be explained in detail. Let me first take you through the story of how it all happened.

B. AdS/CMT: doping the densely entangled holographic CFT.

Let us zoom in on finite density holography. The point of departure is the zero density "Maldacena" state that I highlighted in the previous section. As I explained, this is a zero rest mass Lorentz invariant system like graphene at charge neutrality characterized by "stoquastic quantum supremacy" as testified by the tunable anomalous dimensions. The dictionary spells out what is required in the bulk in order to raise the chemical potential in the boundary to obtain a *finite* density in the boundary: one has to accommodate an electrical monopole charge in the deep interior. This is like raising the chemical potential in a strongly interacting critical incarnation of graphene physics. The outcome is a state that is obviously "infested by fermion signs" and since the zero density matter is already densely many body entangled one is not surprised that the finite density metal is also quantum supreme.

The striking part that fuelled to quite a degree the rapid development of this "AdS/CMT" affair in the string theory community in the period 2007-2013 is that with the eyes half closed the gross physics of these strange metals is quite similar as to that of Fermi-liquids. Initially it was surely a rather subconscious affair: by just experimenting around with finite density AdS/CFT, the "black holes" produced signals that were somehow familiar. In fact, this development accelerated by a development I was myself directly involved: the holographic "photoemission" showing the presence of Fermi-surfaces [13]. Although it became clear later that this is one of the occasions ruled by large N UV sensitivity it stressed the similarity with the familiar condensed matter fermiology (Section VI E 4).

It took myself actually quite a while before I became fully aware of the origin of this shared intuition. The revelation is that the gross structure of the holographic strange metal phenomenology is the same as that of a Fermi-liquid as I just announced. I became fully aware of the "covariant" scaling structure that appears to be realized first by Gouteraux only rather recently [**?**]. Let me first take you through the historical development that culminated in the present understanding.

C. The first steps: the Reissner-Nordstrom strange metal.

As I announced, I will not elucidate the detailed workings of the holographic duality – the gross "landscape" as sketched in the previous section should suffice to give you a sense of orientation and I will just state the various dictionary entries that are required, and sketch what is going on in the gravitational dual. You can look up in the textbooks [9, 10] how it really works. Once again, it is a tight mathematical framework: solve the Einstein equations, unleash the dictionary and there is no room to tamper. This is not like the "scenario" theories that are a mainstream in this branch of condensed matter physics.

The plain-vanilla correspondence that started with Maldacena is at the end of the day about the CFT's, as associated with "bosonic" quantum phase transitions that occur at zero density. What has the dictionary to say about *finite* density? The answer is very simple: an electromagnetic monopole electrical charge has to be accommodated in the deep interior of AdS. The dictionary spells out that the electrical field lines emanating from this charge when they pierce through the boundary are associated with the charge density in the latter. The chemical potential of the boundary becomes finite.

But what kind of stuff is sourced by this chemical potential? This is a tricky affair, and I have perceived it myself all along as one of these items where one has to be prepared for "UV sensitivity". It will become clear soon that this finite density holography is revealing "strong emergence": the whole is so different from the parts that the nature of the parts may no longer matter at all. It better be so because the UV stuff of holography (large N Yang-Mills, etcetera) has no relation whatsoever with the Schroedinger equation "chemistry" of the electrons on the lattice scale. The quote "UV sensitivity" refers to a breach in this central principle. Especially symmetry related affairs hard wired in the UV may turn the IR to be special, and thereby no longer generic. At several instances we will meet this in the remainder.

Surely, at zero density there are no particles to count: it is the "unparticle" CFT quantum soup. How to think about quantum statistics? In the minimal "set up" departing from Maldacena's large N CFT we do know that the boundary is controlled by maximal supersymmetry where the bosonic and fermionic fields are in a perfect balance, with the consequence that no fine tuning of coupling constants is required for conformal invariance.

In addition, it is also understood that this is pertaining to fields belonging to the adjoint of the YM theory: this refers to the force carrying "gluons" that because of the supersymmetry appear as bosonic- and fermionic incarnations that are in a perfect balans. The quarks are "in the fundamental": these can be incorporated in holography as well but this involves more fanciful "brain intersection" holographic constructions [8]. I will largely ignore these: it involves constructions of a higher mathematical sophistication describing a more complex physics, while up to now I have not quite seen outcomes that shed more light on the relations with experiment. This is of course not a really good reason but it is just the state of the field.

Supersymmetry is very fragile and the most brutal way to unbalance the fermions and bosons is by going to finite density. But what is "pulled in" by the chemical potential? I never got a clear answer to this question. Just by looking to the outcomes it just appears that

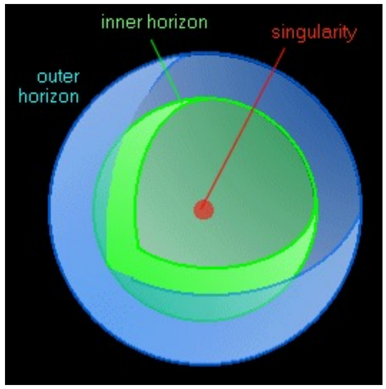

FIG. 10. The "finite temperature" electrically charged Reissner-Nordstrom black hole with its double horizon structure.

"there are a lot of fermions". Perhaps the best evidence is in the form of proof of principle for a Fermi-liquid to appear as an instability of the strange metal – I will come back to this still incompletely understood electron star affair in the next section. But what happened with all these bosonic fields that are present at zero density? It may be that this "doping of supersymmetric stuff" may go hand in hand with an undesired "UV sensitivity". I perceive this as the number one reason to never trust completely the finite density holographic results.

So we have to accommodate an electrical charge in the deep interior. The simplest possible setting to construct a gravitational way to accomplish this is by departing from an Einstein-Maxwell bulk action. This is just the familiar GR textbook affair of considering a universe characterized by gravity (GR, "Einstein theory") and the presence of electromagnetism (Maxwell). There is then one unique outcome that is stationary: the "classic" charged black hole, already solved in the 1920's by Reissner and Nordstrom (RN).

This is part of the basic GR repertoire as discussed in textbooks. While it is still an elementary exercise to derive the solutions such RN black holes have quite a bit more structure than the elementary Schwarzschild version (Fig. 10). These have *two* horizons: an inner- and outer one. Upon passing the outer horizon the geometry switches from time-like to space-like, flipping back to a time-like inner region, implying that the singularity can be avoided. It even turns out to repel observers with a finite rest mass. But there are more puzzling features that appear to still confuse even the specialists. The inner horizon turns out to be a Cauchy horizon which means that it is not possible to extend a geodesic through this horizon. In addition there is a famous conundrum called "mass inflation": a debate has been raging forever whether such black holes can be stable at all in the presence of in-falling energy.

But these difficulties are avoided in AdS/CFT – the boundary has no knowledge regarding what happens behind the horizon. The story continues that a limit can be reached where the energy of the black hole coincides with the electromagnetic energy: the "extremal" RN black hole. The two horizons merge in a double coordinate singularity and the surface gravity is vanishing. A principle of black hole radiation is that the Hawking temperature is set by this surface gravity and it follows immediately that the extremal BH encodes for a *zero* temperature, finite density state of the boundary matter.

But the horizon area is still finite, and although the Hawking temperature vanishes there is therefore still entropy set by the area of the horizon according to the Bekenstein-Hawking entropy governing the boundary field theory. The RN strange metal is characterized by *zero temperature* entropy and this is a pathological feature. In chemistry the fact that zero temperature entropy is not admissible in physical systems is sometimes called the "third law of thermodynamics". But this is overstated: it is a pragmatic affair, any small influence that will be typically overlooked will have the effect of lifting the ground state degeneracy.

However, in the big portfolio of top-down holographic set ups that are derived from "first principle" string theoretical constructions such RN solutions do not occur. There are always other fields present beyond Einstein-Maxwell and these take care that extremal RN black holes with their zero temperature entropy are avoided – see underneath. The RN strange metals are therefore by default pathological. But they are simple and should be looked at as toy models that do grab some of the generic features of the big family of holographic strange metals.

The early results on the RN metal did however reveal already the big picture of the grand principle behind the holographic strange metals. Moreover, the outcomes directly caught my full attention since these are already quite suggestive regarding a major puzzle that seemed to be revealed already in the 1990's in experiments : the "local quantum criticality" as I will explain in a moment. Hong Liu and coworkers from MIT deserve the full credit for finding out how this can be deduced from the bulk resting on the notion of geometrizing the renormalization group in terms of the isometry of the bulk geometry [43, 44].

I already advertised that charged black holes associated with finite density in the boundary have a geometrical structure that is very different from the Schwarzschild black holes encoding for finite temperature at zero density. These imprint on AdS in the following way. Near the boundary it is business as usual but upon descending towards the interior the geometry completely reorganizes upon crossing the radial coordinate associated with the chemical potential μ in the boundary. This turns out to be the way that the "Fermi degeneracy" is encoded in the bulk: μ has a role similar to E_F in the boundary as we will see.

Upon descending further along the radial direction the geometry crosses over rapidly to the *near-horizon* geometry of the charged black hole. For the plain-vanilla RN black hole the metric is textbook material, but the near horizon geometry is quite remarkable. It is called "$\mathrm{AdS}_2 \times \mathrm{R}^d$". R^d refers to the space dimensions shared with the boundary, and it just refers to the fact that it is a flat space. However, the remaining time- and radial direction morph in a two dimensional Anti-de-Sitter: the AdS_2. Focussing in on the zero temperature extremal case, the geometry is completely different from the pristine near boundary AdS geometry encoding for the zero density stoquastic CFT. This is the bulk encoding of the strong emergence hard wired in finite density holography: the scaling properties of the finite density state at scales small compared to μ are entirely unrelated to those of the zero density stoquastic type CFT.

Given the "GR = RG" principle, "AdS" means that in the boundary one will find a CFT like scaling behaviour but in the RN AdS_2 "throat" (referring to the geometry close to the horizon) this only acts out in the plane spanned by time and the radial direction. The radial direction is just the scaling direction and henceforth this implies a purely *temporal* scale invariance. Let us recall the definition of the dynamical critical exponent: $t \sim l^z$, where t is time and l is length. We are now dealing with a system where the correlation time $\tau_{\mathrm{cor}} \to \infty$, it is temporally critical. But the spatial correlation length is somehow finite (see Section VII D 1). This implies that the dynamical critical exponent $z \to \infty$!

As I will discuss in Section (IX B 3), such a kind of a behaviour was inferred from experiments on the cuprate strange metals already in the late 1980's. It is a cornerstone of the early "marginal Fermi-liquid" phenomenology [45], while it was observed in dynamical experiments more recently. It got the name "local quantum criticality" and this semantics was directly used in holography as well.

This was all along a great mystery. In Section (III) I explained that one may encounter a z of two or three at least in the absence of strong quenched disorder but an infinite z appears to be excluded from this agenda. There is a claim that it can happen in certain dissipative settings [46] but this is quite controversial. When I learned about this infinite z arising in such a natural but unfamiliar fashion I got immediately greatly intrigued, embarking definitively on the risky pursuit of concentrating my research entirely on holography.

But in these early days there was also quite a bit of confusion. There was initially a perception that the RN strange metal was the unique finite density metallic state predicted by holography. There was the uneasy aspect of the zero temperature entropy – at a point there were even false claims originating in the experimentalist's community that such zero T entropy was observed. Until the present day you may find outsiders who are claiming that holography is all wrong since it predicts such entropy. This is nonsense – we understand this much better now: the dilatons change the rules as you will see now.

D. The dilatons, hyperscaling violation scaling and the fermionic degeneracy scale.

To make this "symmetry processing power" of holography to work one has to make sure that the bulk gravitational theory is sufficiently general to represent the universal features of the boundary phenomenology. In this regard the minimal Einstein-Maxwell gravity that predicts the Reissner-Nordstrom black hole to be the unique gravitational dual of the boundary metal is falling short. Once again, the string theorists have mighty mathematical machinery in the offering in the form of the "top-down" holographic set ups. As the original Maldacena construction is to physicists standards mathematically proven to express a precise duality, they constructed subsequently a plethora of such "exact" dualities involving richer physical circumstances. It is all large N limit and these translate therefore to classical gravitational physics in the bulk, involving fields that may be rather unfamiliar. But one continues to discern "Einstein theory universality". These extra fields hard wire yet other universal traits in the bulk that translate into universal properties of the boundary phenomenology. One can let again the numbers to be pending the specific UV while the structure of the deep IR theory may have a much greater applicability.

1. Kaluza-Klein compactification and the dilatons.

Turning to finite density, the top-down "set ups" are univocal: even for the relatively humble affair of determining the structure of the thermodynamics in the boundary one *has* to take care of an additional field, the *dilaton*. Dilatons may be unfamiliar but they are bread and butter in string theory. Kaluza-Klein compactification is fundamental to all of string theory including the correspondence. This goes back to the demonstration in the 1920's by Kaluza and Klein that when one departs from pure GR in 5 space time dimensions, upon rolling up one of the space dimensions in a circle one obtains Einstein-Maxwell theory in 4 dimensions. Fundamental string theory can only be formulated in 10 overall dimensions, while the specific limit taken in holography typically involves classical supergravity in 11 dimensions. One has to roll up, say, 7 of these dimensions to obtain the 4 dimensional bulk gravity required to encode the boundary theory in 2+1 dimensions. In these high dimensional circumstances there are many ways to accomplish this rolling up in compact dimensions (the Calabi-Yau manifolds) and pending what one picks one arrives at different lower dimensional theories.

However, a common denominator is that regardless how one compactifies a new scalar field drops out the Kaluza-Klein procedure that has the typical effect in Einstein theory to affect the volume of space: it is therefore called the "dilaton field". To give an impression of how the action of such a "Einstein-Maxwell-Dilaton" theory looks like,

$$S = \frac{1}{16\pi G}\int d^{d+2}x\left[(\mathcal{R} - 2(\nabla\Phi)^2 - \frac{V(\Phi)}{L^2}) - \frac{Z(\Phi)}{4e^2}F_{\mu\nu}F^{\mu\nu}\right] \tag{48}$$

where G, $\mathcal{R}$ and L are Newton's constant, the scalar curvature and the AdS radius respectively. $F_{\mu\nu}$ is the Maxwell field strength and e the charge. The novelty is in the scalar field Φ: the dilaton. In the deep interior Φ becomes typically large and the potentials may acquire the odd-looking generic forms like $Z(\Phi) = Z_0\exp(\alpha\Phi), V(\Phi) = -V_0\exp(\beta\Phi)$. The structure of these potentials are generic and these give rise to various general circumstances in the boundary. One possibility is that the deep interior geometry just "disappears" and these (soft-, hard-) walls describe *confining* states in the boundary: these flourished in the context of the "AdS/QCD" pursuit [8, 9] . However, given the form of the potentials as I just quoted, as function of the free parameters like α, β these describe a vast family of "near horizon scaling geometries" that dualize in a *family of scaling theories describing* *finite density matter.*

2. The near horizon scaling geometries of EMD gravity.

Gouteraux, Kiritsis and coworkers [47] demonstrated in their seminal work that EMD gravity can be used to classify. A seemingly universal scaling theory for the finite density matter can be extracted, having a similar status as Kadanoff's scaling theory for the conventional critical state of Section (III B 3). This hinges on the GR = RG principle: derive the geometry in the deep interior resting on universal characteristics of the gravitational theory, and when this turns out to be a "scaling geometry" (as is the case for EMD gravity) it will dualize in a scaling prescription in the boundary.

Following this strategy they show that the deep interior metric acquires the general form,

$$ds^2_{\text{EMD}} = \frac{1}{r^2}\left(-\frac{dt^2}{r^{2d(z-1)(d-\theta)}} + r^{2\theta/(d-\theta)}dr^2 + dx_i^2\right) \tag{49}$$

where t, r, x_i are (Lorentzian) time, radial coordinate and space coordinates, respectively. The metric is responsible for thermodynamics and it has an isometry that translates into scaling laws in the boundary in terms of two free parameters θ and z (d is the number of space dimensions, as usual). In the top-down like setting these are determined by the specifics of the dilaton potentials. One is not surprised that in the boundary dual z corresponds with the now familiar dynamical critical exponent. But pending the dilaton potentials this can now take any value $1 \leq z < \infty$. The Lorentz invariant (zero density) and local quantum critical scaling behaviours are just the extremal cases. The news is in θ, called the "hyperscaling violation exponent". It is called like this because the free energy is no longer scaling with the volume of the system $\sim L^d$ but instead with a "reduced" volume $L^{d-\theta}$. It can take all values $d < \theta < -\infty$ without running into no go theorems – the precise conditions are $(d-\theta)/z \geq 0, (d-\theta)(dz-d-\theta) \geq 0$ and $(z-1)(d+z-\theta) \geq 0$

The simplest consequence for the boundary theory is that the entropy (or equivalently the specific heat $C \sim T(\partial S/\partial T)$) exhibits the following scaling behaviour,

$$S \sim T^{(d-\theta)/z} \tag{50}$$

Dealing with the zero density state the scale *invariance* of the free energy imposes that $\theta = 0$. For a Lorentz invariant theory ($z = 1$) one recovers the "Debye" entropy $S \sim T^d$

that is indeed generic for all conformal field theories. Dealing with the $z \neq 1$ zero density theories it is well known that the entropy $S \sim T^{d/z}$; the factor z is there to correct for the "number of time dimensions" when the Euclidean time direction is rolled up in the thermal circle.

However, the hyperscaling dimension θ is unfamiliar in this zero density repertoire. You may not have realized it yet but it is overly familiar in a different context: it is the "central scaling gear" controlling the algebraic properties of the Fermi-liquid! The dimension θ controls how the number of *thermodynamically relevant* degrees of freedom scales with the volume of the system. In the Fermi liquid these are controlled by the Fermi-surface having a dimensionality $\theta = d - 1$. At every point on the Fermi-surface excitations are anchored with a linear dispersion: $z = 1$. Fill these values into Eq. (50) and you find $S \sim T$ regardless the number of space dimensions d. But this is the overly familiar Sommerfeld entropy of the Fermi liquid which scales in the same simple way regardless the number of space dimensions!

3. Finite density and the covariant renormalization flows.

What is going on here? As I already emphasized in Section (V B) the deep mathematical underpinning of holography is in the relation between the *isometry* of the geometry in the bulk and the *symmetry* controlling the boundary field theory. For the GR illiterate, isometry is a property that one encounters dealing with curved space times. For instance take the two dimensional surface of a ball. This manifold is curved everywhere but in this regard every point on the surface is characterized by the same curvature. This is an example of maximal isometry.

From a CM perspective the interest is in *emergent* symmetries like the conformal invariance that may be realized at a stoquastic quantum critical point. Right now we are staring at a different type of symmetry at work in the boundary system that is about a particular way the system is changing continuously under scale transformation that we recognize to be already at work in the Fermi-liquid.

Let me stress it again: the isometry-symmetry relation is at the core of the RG=GR "miracle" of holography. The curved geometry in the bulk *geometrizes* the RG flow, the isometries in the bulk as they relate to the radial "scaling" direction paint a picture of the RG flows that can be digested by our visual system when trained to "see" the curvature.

Given the success of RG = GR at zero density, revealing that the ensuing holographic RG flow is way more universal than the special large N CFT suggests that a similar "hyper universality" may also be at work at finite density. I perceive this as the 64K$ question in this affair.

To understand in what regard the isometries of finite density holography are different from those of the zero density CFT's let us zoom in on how exactly the bulk geometry depends on *scale transformations*. It is actually quite simple – to follow the arguments you only need a minimum of GR background. Let us start with the simplest of all geometries – Minkowski flat space-time. The invariant is the metric, in Cartesian coordinates $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$, where μ, ν refer to spacetime directions ($\eta_{tt} = -1, \eta_{aa} = 1$, a referring to space dimensions).

Your intuition may give you the impression that flat space-time should be independent of scale. It does transform in a smooth, continuous way under scale transformations but it is actually not *invariant* under scale transformation. This is easy to find out. Perform a scale transformation on the coordinates: $x^\mu \to \Lambda x^\mu$ and it follows immediately that $ds^2 \to \Lambda^2 ds^2$. The metric is proportional to itself under scale transformation but it does not continue to be the *same* metric. The geometer refers to this as "the metric transforms *covariantly* under scale transformation".

The CFT at zero density is however *invariant* under scale transformation. The "isometry = symmetry" principle now insists that the metric in the bulk has to be scale *invariant* as well, covariant is not symmetric enough. Such a geometry is actually a necessary condition for e.g. the exercise that I outlined in Section (V B) where the mass of the bulk fields turns into the scaling dimension of the boundary branch cut to work.

Mathematical precision kicks in: there is just one unique metric associated with Riemannian manifolds that has the property of invariance under scale transformation. This is Anti-de-Sitter space. Choosing a radial coordinate r that varies from 0 in the deep interior to ∞ at the boundary and normalizing by choosing for the AdS radius $L = 1$, the AdS metric can be written as,

$$ds^2_{\mathrm{AdS}} = \frac{1}{r^2}\left(\eta_{\mu\nu} dx^\mu dx^\nu + dr^2\right) \tag{51}$$

It s very easy to check that under the scale transformation $x^\mu \to \Lambda x^\mu, r \to \Lambda r$ the metric $ds^2_{\mathrm{AdS}} \to ds^2_{\mathrm{AdS}}$. The metric is invariant: the conclusion is that the RG flow of a CFT is *uniquely geometrized by the AdS space*!

Let us now inspect how the finite density scaling geometry family Eq. (49) behaves under scale transformation. As in the boundary, we depart from a redefinition of the spatial scale, $x_i \to \Lambda x_i$. You are now used to the meaning of the dynamical critical exponents, $t \to \Lambda^z t$. You then discover that the radial coordinate will transform according to $r \to \Lambda^{(d-\theta)/d} r$. For $z = 1, \theta = 0$ one recovers the scale invariance – Eq. (49) reduces to pure AdS, Eq. (51). Insert these rescalings in Eq. (49) an you will find,

$$ds^2_{\text{EMD}} \to \Lambda^{2\theta/d} ds^2_{\text{EMD}} \tag{52}$$

We have uncovered the essence of the EMD scaling geometries. When $\theta \neq 0$ the structure of the RG flows is governed by *covariance* under scale transformation instead of the invariance underpinning the scaling relations of CFT's. Blaise Gouteraux appears to be the first who recognized this crucial insight (see e.g. Ref. [12]).

This has various consequences. A first consequence is that in a covariant flow the UV scale where it starts is remembered in the deep IR. In a scale invariant flow the information regarding the UV scales completely disappears. This is the usual affair of UV regularization – the IR theory is strictly independent of the short distance physics. But in the covariant flow the metric is changing via the proportionality factor associated with the scale change. This dualizes in the boundary to the phenomenon that the IR observables still know about the UV point of departure. This is of course the "degeneracy scale", the Fermi-energy in the Fermi liquid that just acquires the more agnostic name "chemical potential" in holography. Dealing with strong emergence as in condensed matter this appears to be also linked to the fact that conformal invariance can only arise by infinite fine tuning to a quantum critical point, while for covariant scaling behaviour there is plenty of room for *phases* of matter that do not require anything of the kind. The Fermi-liquid is case in point.

A next big difference is that to be only *covariant* instead of invariant under scale transformation means that this symmetry is much less powerful in constraining the behaviour of observables. As will become clear when I further unfold the rather rich portfolio of physical phenomena of the finite density holographic systems, much more is going on than in the zero density CFT's going way beyond the still rather simple extension of the thermodynamics embodied by the entropy, Eq. (50). Whether this implies that the IR physics becomes more susceptible for "UV dependence pollution" is a next issue – there are reasons for concern, see underneath.

Although the geometry Eq. (49) was discovered by solving concrete, top-down inspired holographic set ups one may actually turn it around. Dealing with the CFT's one could depart from the conformal symmetry of the boundary theory, to then discover that the only form of geometrized RG flow is in the form of the pure AdS geometry. Dealing with finite density, one can depart from the scaling structure of (say) the Fermi-liquid, in addition to be enlightened regarding the possibility of a $z \neq 1$, to then pose the question what kind of isometry can reconstruct this behaviour? Relying on the simple scaling exercise in the above, one will then concludes that the metric Eq. (49) is the unique outcome in the same guise that pure AdS is the unique geometrization of the CFT RG flow.

Should it be for this reason that Eq. (49) represents the truly universal "library" of RG flows? This appears to be a quite defendable conjecture dealing with *homogeneous* geometries. However, as I will discuss at length in the next two sections one has to cope with a very different side of gravity in case that the space is not homogeneous and/or isotropic. Einstein theory shows here its real face in the form of a system of non-linear partial differential equations and it becomes very hard if not possible to write down transparent, closed solutions like Eq. (49). Electrons in solids fall in this "inhomogeneous" category and it may well be that yet different scaling geometries arise that may be more complex given that symmetry exerts less control. Presently, nobody has an answer to this question and it is a main challenge of our numerical holography effort in Leiden.

E. The holographic strange metals as generalized Fermi-liquids.

I just announced that the phenomenology controlled by the "covariant" scaling is quite a bit richer than for the CFT's. In fact, it is a vast landscape that is not at all completely explored. One can yet identify a number of general features. The overarching message is that the gross "organization" of such metallic states rests on a template formed by the Fermi-liquid, which is then extended by turning the scaling dimensions to become anomalous relative to the "engineering" dimensions of the Fermi-liquid. Let me put some meat on the bare bones, focussing on particular aspects of this phenomenology. A lot more is known and I refer to the textbooks for a more exhaustive discussion.

1. *The thermodynamics: embarrassing the marginal Fermi-liquid.*

If anything stands a chance to be truly universal it is the simple but very powerful formula Eq. (50) for the entropy. I already unveiled that it captures the scaling of the thermodynamics of the Fermi-liquid in an extremely efficient fashion. There is no need to hassle with the integrals involving Fermi-Dirac distributions as is done in the elementary textbooks. In all its simplicity, this formula can be used to great effect in the empirical arena of cuprate strange metals. This is really the subject of the last section but this is so elementary that it deserves to be included already at this point if not only because it illustrates the amazing powers of "covariant RG flow".

One immediately infers that by assuming $d - \theta$ to be finite it follows immediately that $S \sim T^0$ in a local ($z \to \infty$) quantum critical system: the temperature independence implies that there is zero temperature entropy. This is precisely the origin of its $T = 0$ entropy pathology of the Reissner-Nordstrom strange metal that of course also submits to the scaling logic.

But as I will discuss in detail in Section (IX B 3) there is direct experimental evidence supplied by electron loss spectroscopy for local quantum criticality in the cuprate strange metals. This was actually already the central pillar of the very early "marginal Fermi-liquid" theory [45]. This got later a more precise identity within the Hertz-Millis scheme of Section (III D): the Fermi-liquid quasiparticles decay in the continuum of local quantum critical fluctuations associated with a QCP by a Yukawa coupling [28]. By involving in essence second order perturbation theory, one then finds that the entropy coming from the quasiparticles diverges logarithmically through their effective mass.

However, in this mindset it is completely worked under the rug that this critical continuum represents also thermodynamically relevant degrees of freedom! Since $z \to \infty$ it appears to be that the marginal Fermi liquid entropy is temperature independent! The electronic specific heat in the strange metal regime was measured a long time ago and it appears to be Sommerfeld $S \sim T$ – initially it was without further thought assumed to reflect a Fermi liquid. This actually amounts to a no-go theorem for theorem for this marginal Fermi-liquid affair, in fact largely overlooked until the present day.

But there appears to be still a problem of principle in the cuprate metal context: how to reconcile $z \to \infty$ with a Sommerfeld entropy? Top-down holography offers here a remarkable solution, that is also quite elegant viewed from the string theoretical side. It was actually introduced in the AdS/CMT context by Steve Gubser [48], the G of the GKPW dictionary, who unfortunately deceased at a young age through a mountaineering accident.

This top down involves a RN extremal black hole in 11 dimensional supergravity that after compactification just "touches" the space plane of the 4D holographic bulk in such a way that the horizon just shields the singularity while the area of the horizon becomes zero at zero temperature. This has therefore zero temperature but one finds that besides $z \rightarrow \infty$ also $\theta \rightarrow -\infty$ such that the ratio $-\theta/z \rightarrow 1$: this results in a Sommerfeld entropy $S \sim T$!

Although I have no clue what $\theta \rightarrow -\infty$ means, the fact that this "conformal to AdS2" metal is top down consistent means that it can arise in a physical theory. So much is clear that the cuprate strange metal cannot possibly controlled by the $\theta = d - 1$ Fermi-surface hyperscaling violation. Although the whole condensed matter community has been taken for granted that Fermi-surfaces exist in strange metals, on closer consideration thermodynamics excludes directly this possibility.

2. The generalized Lindhard excitation spectrum of strange metals.

The spectrum of charge excitations is a first property to look for – it can be measured by electron-loss spectroscopy (Section IX B 3) and it figures prominently in the textbook treatment of the Fermi-liquid. At stake is the charge density - charge density propagator or dynamical charge susceptibility. One is interested in the spectral function – the imaginary part – in a large kinematical range of energy, temperature and momentum.

Once again, this is canonical in the Fermi-liquid. One departs from the non-interacting Fermi-gas in the Galilean continuum. The dynamical charge susceptibility/polarization propagator $\chi_0(\mathbf{q}, \omega)$ just counts the number of electron-hole pairs that can be created at energy ω and total momentum $\mathbf{q}$. This is the overly well known Lindhard function (Fig. 11) with its characteristic bounded spectrum in the momentum-energy plane. As I already discussed, the frequency dependence is an algebraic affair at larger momenta: $\mathrm{Im}\chi_0(\mathbf{q}, \omega) \sim \omega$ in $d = 3$. But the long wavelength density fluctuations are subjected to hydrodynamical protection. Since total momentum is conserved in homogeneous space there has to be a protected propagating sound mode.

The fixed point physics is captured by time dependent "classical" mean field known as

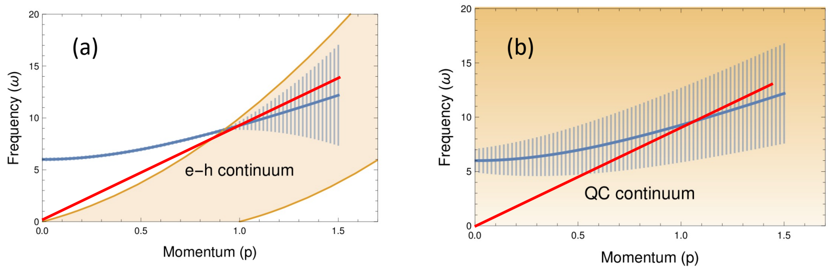

FIG. 11. The charge response. In (a) the Lindhard spectrum formed by the electron-hole excitations around the Fermi-surface is indicated. In addition, the sepctrum is characterized byzero sound (red line) that turns into the plasmon in the charged Fermi-liquid [49]. The response in a $z \to \infty$ holographic strange metal is characterized also by the zero sound/plasmon but the Lindhard continuum is supplanted by a continuum characterized by anomalous scaling dimensions.

"Random Phase Approximation" or "bubble resummation". In the presence of an instantaneous repulsive density-density interactions V_q the $T = 0$ charge susceptibility becomes,

$$\chi(\mathbf{q}, \omega) = \frac{\chi_0(\mathbf{q}, \omega)}{1 - V_q \chi_0(\mathbf{q}, \omega)} \tag{53}$$

For a short range interaction, one finds that an "antibound" propagating mode emerges with a linear dispersion that is lying above the upper bound of the Lindhard continuum. This is the zero sound, having the interpretation that the Fermi-surface turns into an elastic-like manifold that supports coherent vibrations. Via the Luttinger-volume theorem this in turn corresponds with a longitudinal density oscillation – sound. This is promoted to the plasmon dealing with the long range Coulomb interaction. One ascends from the fixed point by perturbation theory (e.g., use dressed fermion lines in the bubble) rendering e.g. an attenuation at finite momentum that is diffusional $\sim q^2$.

As I already emphasized, one can adiabatically continue the finite temperature Fermi-liquid at sufficiently long times all the way to the high temperature limit and it is therefore macroscopically governed by Navier-Stokes – e.g., the viscosity becomes very large at low temperature, $\eta \sim (E_F/k_B T)^2$. But it also carries the "collision-full" first (normal) sound crossing over to "collisionless" zero sound at $\hbar\omega \simeq k_B T$, with a spectacular change in attenuation mechanism. All of this got triumphantly confirmed in ^{3}He.

When we turn to transport later, one should be aware that generically the long wave length currents (and transport coefficients) are actually directly tied to the sound mode. This is hard wired in standard linear response: via the continuity equation $\partial_t \rho + \vec{\nabla} \cdot \vec{J} = 0$ rooted in charge (number) conservation the density and current reponses are tied together. A sound mode that lives infinitely long at infinite wave length as protected by translational invariance in a finite density system translates into perfect conduction.

Highlighting the interpretation of the holographic strange metals as generalized Fermi-liquids, the charge excitation spectrum of the former is organized in the same way as the latter, see Fig. (11) and e.g. Ref. [49]. Also in holographic metals one finds zero temperature sound turning into first sound at finite temperatures. This is perhaps unavoidable: momentum and charge conservation imply the hydrodynamically protected mode. In addition, the holographic strange metals also exhibit a spectrum of incoherent excitations (like the Lindhard continuum) characterized by power-law behaviour. But the scaling dimensions are now set by the anomalous dimensions!

In fact, next to the thermodynamic scaling dimensions θ and z that I discussed in the above, dealing with charge related reponses there is room for one more anomalous dimension in the EMD bulk: the "charge exponent" ζ [47]. This is expressing that the electrical charge is running under RG. This is a bit of a delicate affair. The electric charge quantum is *locally* conserved, "protected by Gauge invariance", and is not supposed to run. This has the ramification that the electrical charge quantum can be deduced from macroscopic measurements; e.g. there is a factor $2e$ in the Ginzburg-Landau free energy that becomes observable by measuring the quantized magnetic flux associated with a superconducting vortex. To see ζ at work one has to lift the charge conservation. This has then intriguing consequences for instance for the Aharonov-Bohm effect as analyzed by Phillips and coworkers [50], next to modifying the scaling behaviour of the charge responses. Given that such exotic physics may be less relevant in the normal state of cuprates, and to keep things as simple as possible I will ignore here ζ – its effects can be easily retrieved from the literature.

We now arrive at the bottom line. As in the Fermi-liquid, the sound- and incoherent sectors occur as parallel channels: $\chi(\mathbf{q},\omega) = \chi_{\rm D}(\mathbf{q},\omega) + \chi_{\rm inc}(\mathbf{q},\omega)$. The D refers to "Drude", the conventional name referring to the (nearly) hydrodynamical sector. It is often called "coherent" but this is a bad name because it has nothing to do with wave coherence, that is just semantic confusion originating in Fermi-gas pathology where one departs from free fermion quantum mechanical coherence. The "other stuff" I will continue to call "incoherent" although it really refers to "unparticle stuff" that is of course many body quantum "coherent" at zero temperature.

The Drude part is qualitatively the same affair as in a Fermi liquid while the big difference is in the incoherent part showing E/T type scaling but now modified by the anomalous dimensions. By mere scaling reasoning it is straightforward to find out that for $\omega << \mu$ and $\omega > T$ the spectral function scales like,

$$\mathrm{Im}\chi_{\rm inc}(\mathbf{q},\omega) = \omega^{(d-2-\theta+z)/z}\mathrm{F}\left(\frac{\omega}{|q|^z}\right) \tag{54}$$

Let us give a check for the Fermi liquid in $d = 3$: as before $\theta = d - 1$ (Fermi surface) and $z = 1$, while the scaling function relating frequency and momentum $F(\omega/q^1) = \omega/q$: the formula recovers the well stablished scaling behavior $\chi" \sim \omega/q$ of the Lindhard continuum at energies small compared to E_F. This is quite remarkable: the scaling of the charge excitations is determined by the thermodynamics (θ, z) in a way that is entirely unrelated to the way the correlation function exponents are tied to thermodynamic scaling dimensions at stoquastic quantum critical points.

The scaling function F(x) is obviously important but we know little about it – it is just not computed yet in EMD backgrounds. This becomes especially a delicate affair in local quantum critical systems, $z \to \infty$. In this limit, from mere scaling consideration one just learns that the correlation time is diverging while the correlation length may be finite. This has only been looked at in the simple RN throat. Remarkable things happen here. Resting on the causality structure of the near horizon geometry Hong Liu and coworkers [51] showed that this is ruled by a length $\xi \simeq \mu$. At smaller lengths the system is quite incompressible while the charge susceptibility scales in an odd way with the scaling dimension pending momentum, $\mathrm{Im}\chi(q,\omega) \sim \omega^{2\nu_q - 1}$ where $\nu_q = (1/2)\sqrt{5 + 4q^2 - 4\sqrt{1 + 2q^2}}$ in $d = 2$. At scales larger than ξ one is dealing with patches that are not in causal contact in the bulk, meaning that these fluctuate independently in the boundary. In this regime the response is dominated by the hydrodynamical Drude affair..

At a finite temperature the incoherent responses just submit to energy-temperature scaling: Im$\chi \sim \omega^\alpha$ when $\omega >> T$ turns into Im$\chi \sim T^\alpha$ when $\omega << T$. In the Drude sector one observes similarly the crossover from zero- to hydrodynamical first sound. Also in this regard the Fermi-liquid yields the template for the generalization to the strange metal. Yet the dissipation in the thermal fluid is very different from the Fermi-liquid, now submitting to the Planckian dissipation dimensions.

3. Strange metals turning into superconductors: "quantum critical BCS".

The first result signalling that AdS/CFT could have dealings with electrons in solids was the discovery of holographic superconductivity, independently by Hartnoll, Herzog and Horowitz ("H^3") and Gubser in 2008, see the books. This actually triggered the AdS/CMT frenzy that followed in the string theory community.

Gravitationally this is a remarkable affair. Everybody has heard of course about the "no-hair theorem". It is mathematically proven that stationary black holes cannot carry detailed traits: they are like elementary particles, completely characterized by overall quantities including their mass, charge, linear- and angular momentum. But this theorem depends critically on flat asymptotics: there is no no-hair theorem dealing with AdS asymptotics!

A general way to capture spontaneous symmetry breaking is implied by linear response theory: when there is a response without a source one is dealing with an ordered state. In Section (V) I sketched the way that this dualizes via the dictionary to the bulk finding out that the source and response are associated with the leading- and subleading components of the bulk fields asymptoting near the boundary. One then finds out that the VEV without a source can only happen when the bulk field acquires a finite amplitude in the bulk, in essence describing that the black hole acquires an atmosphere formed from this stuff. It is a rather unusual atmosphere as compared to what one encounters in the cosmos: it extends all the way to infinity along the radial coordinate, with a density falling off with a power law.

According to the dictionary the "breaking of the $U(1)$" associated with superconductivity dualizes in a complex scalar field in the bulk. Now it comes: the (positive) mass-squared of such a bulk field will in first instance code for the scaling dimension in the boundary. However, when one inserts such a field in the strongly curved background near the horizon, one may run into the "BF bound": this is the gravitational phenomenon that the curvature may translate into the mass-squared turning negative in a flat manifold: this is the usual mexican hat affair. The outcome is that upon lowering the temperature in the boundary the near horizon curvature is growing until the BF bound is violated and the black hole acquires a spontaneous finite amplitude "scalar atmosphere" that turns out asymptote near the boundary precisely in the desired fashion!

This is a miracle. In the bulk gravitational machinery is at work that was not realized in the canonical agenda of GR before – the BF bound violation triggering the hair. This is then processed by the dictionary to a boundary physics that is precisely right. This is just like a mathematical clockwork where all the gears work together to produce an ultimately precise action. Given that the correspondence is mercilessly precise with anything that is controlled by symmetry, this holographic superfluid/superconductor (one can effectively gauge the boundary) is impeccably reproducing the phenomenology: one finds perfect conductors, the fluxoids/quantized vortices are reproduced, etcetera. This was even used [52] to shed light on superfluid turbulence in two dimensions! In fact, it became clear later that this is only the tip of the iceberg. In Section (VIII) I will summarize the state of the art of holographic symmetry breaking finding that the most natural and general bulk dualizes into a boundary order that is strikingly similar as the "intertwined" orders observed in underdoped cuprates.

But the Bardeen-Cooper-Schrieffer theory of conventional superconductors goes a step further. It predicts that at T_c a gap opens in the spectrum of Fermi-liquid quasiparticles. This gap is proportional to the amplitude of the order parameter and this is famously *exponentially small* in the UV parameters $\Delta \sim \exp(-E_F/V)$ where V is the attractive interaction. But this instability will occur always, regardless the smallness of Δ: the well known principle that the Fermi-liquid as state of matter is only unstable in an absolute sense in the presence of an attractive interaction.

The take home message is that also in this regard holographic superconductivity is a close cousin of the BCS/Fermi liquid version. In fact, the observable physics is so close to BCS that it can be easily misidentified as a BCS superconductor! To observe the sharp difference unusual experiments have to be implemented which turn out to be very difficult to realize in the lab. Let me explain.

How does BCS work? Looking at the charge response as I just discussed, the incoherent (scaling) response is at long wavelength shrouded by the conservation laws – the hydrodynamical soft mode in the form of sound. But the superconducting order parameter is not conserved and the BCS instability actually rests on the scaling properties of the incoherent continuum, now associated with the "particle-particle channel", instead of the particle-hole Lindhard continuum. Let me remind you of the professional (Gorkov theory) way of handling BCS.

The operator of relevance is the pair operator $b_q^\dagger = \sum_k c^\dagger_{k+q/2,\uparrow} c^\dagger_{-k+q/2,\downarrow}$. The interest is in the dynamical pair-susceptibility/ two point pair propagator, $\chi_p(q,\omega) = \langle b^\dagger b \rangle_{q,\omega}$. We can specialize to $q = 0$ – I will touch later shortly on finite momentum pairing, the "pair density waves" believed to occur in underdoped cuprates in Section (VIII). Hence, $\chi_p(\omega) := \chi_p(q = 0, \omega)$.

BCS is just resting on the "classical" mean field theory. Accordingly, one can employ the RPA/time dependent mean filed logic that is also at work dealing with zero-sound/plasmons in the charge density response. One departs from the pair susceptibility of the Fermi liquid in the absence of the attractive interaction $\chi_p^0(\omega)$ to then switch on an attractive short range interaction with strength g. According to the time dependent mean field,

$$\chi_p(\omega) = \frac{\chi_p^0(\omega)}{1 - g\chi_p^0(\omega)} \tag{55}$$

As for zero sound, one is interested in new poles that can appear associated with the denominator,

$$\text{Re}\chi_p^0(\omega) = 1/g \tag{56}$$

For zero sound this condition is satisfied for the anti-bound states living above the Lindhard continuum. But standard BCS follows from the fact that upon lowering temperature one finds an (overdamped) mode from Eq. (55) that hits zero frequency at the superconducting T_c, signalling that the metallic state becomes linearly unstable. Now it comes: the "Cooper logarithm" is in fact associated with the "covariant" scaling property of the pair susceptibility of the Fermi-gas. Let us just assert that $\chi_p^0(\omega) \sim \omega^{2\Delta_p}$ where Δ_p is the scaling dimension of the pair operator. By just counting it for the non-interacting Fermi gas one finds $\Delta_p = 0$: the scaling dimension of the pair operator is "marginal", the imaginary part does not depend on energy but in the characteristic covariant scaling fashion its magnitude is set by the Fermi energy, $\text{Im}\chi_p^0(\omega) = N_0 = 1/(2E_F)$.

But for the instability criterium Eq. (56) we need the real part of χ_0 at zero frequency. According to Kramers-Kronig the real part becomes at zero temperature in the presence of a lower- (gap Δ) and upper (phonon frequency ω_B) bound,

$$\mathrm{Re}\chi_p^0(\omega = 0, T = 0, \Delta) = \frac{1}{E_F} \int_{\Delta}^{2\omega_B} \frac{d\omega'}{\omega'} = \frac{1}{E_F} \log(\frac{2\omega_B}{\Delta}) \tag{57}$$

The BCS logarithm is just an expression of the marginal scaling property of the pair operator. Inserting this into Eq. (56) yields than the famous gap equation $\Delta = 2\omega_B \exp(-1/\lambda)$ where $\lambda = g/E_F$. Resting on the energy-temperature scaling property of χ_p it is easy to find out how this translates into the expression for T_c.

Explaining the exponential smallness of the order parameter in the coupling g is of course the historic insight of the BCS theory – at the same time, although the gap is exponentially small in the UV parameters g, E_F it indicates an absolute instability crirerium: the Fermi liquid *has* to be taken over by the superconductor eventually, be it at an extremely small temperature.

The shear magnitude of T_c in the cuprates has of course played a (too) important role in the sociology of this field that is until the present known as "high Tc superconductivity". Ironically, 99% of all papers that tried to explain it took for granted that this BCS logic is written in stone. What then remained to be done is to find a reason for g to be very large; it has to be since the superconductor has to outcompete the normal state and the "BCS log" is telling us that the superconductor has in this regard literally only a marginal advantage. Hence, all kinds of superglue's were appointed as holy grail such as spin fluctuations, various kinds of QCP quantum critical fluctuations and so forth.

But given that the whole community appears to agree that there is no sign of Fermi-liquid physics in the normal state, why should the dynamical pair susceptibility of the normal state behave as if it is controlled by a free Fermi gas? This is perhaps the most remarkable article of unsubstantiated belief in this community while it is even not realized that it is actually a wild guess! In the same year that holographic superconductivity was discovered, we pointed out [53] that one can easily generalize BCS in such a way that the good parts are maintained such as the opening of a gap at T_c, at the same time generalizing the phenomenology away from the Fermi gas. We called this "quantum critical BCS". This just departs from the wisdom that one can keep the mean field machinery explained in the above, modifying it by considering a generalization of the pair susceptibility of the normal state, $\chi_p^0(\omega)$. We asserted that it could be a scaling function but now one with an unknown scaling dimension of the pair operator precisely in the guise of the covariant RG,

$$\chi_p^0(\omega) \sim \omega^{2\Delta_p} \tag{58}$$

When $\Delta_p = 0$ this recovers BCS. For the "irrelevant" $\Delta_p > 0$ one needs a quite finite interaction for the instability to occur (e.g., in a semimetal like zero density graphene $2\Delta_p = 1$). However, Δ_p can in principle be negative: the "pair operator is relevant in the deep IR". When one now solves the gap equation one finds out an *algebraic* expression for T_c, instead of the logarithm. Instead of this marginal, fine tuned borderline affair that is at work in the Fermi gas the normal state is now *genuinely unstable* towards superconductivity. One may not be surprised that this implies that a quite humble "pairing glue" now gives rise to a *high* T_c.

A little later we zoomed in on the generic features of holographic superconductivity [54]. The pair susceptibility is easy to compute in a holographic superconductor. As I emphasized repeatedly, the properties of the holographic strange metals are by default set by scaling and this surely applies to the pair susceptibility. We found that superconductivity originating in holographic strange metals precisely reproduces the quantum critical BCS scaling theory!

The anomalous dimension of the pair operator are associated with the scaling dimensions of the scalar field which are adjusted by the mass in the bulk that in a bottom up approach is a free parameter. Different from the charge reponses Eq. (54), this is a genuine independent scaling dimension that is not related to e.g. the thermodynamic dimensions θ and z. However, in natural bulk set ups Δ_p tends to be relevant, suggesting that holographic superconductors are typically high Tc superconductors.

The take home message is simple but striking: this example illustrates the general notion that all one needs to know in order to generalize Fermi-liquids to the domain of quantum supremacy is to promote the "mean-field like" scaling dimensions of the Fermi gas ($\Delta_p = 0$) to anomalous dimensions. Because the gross organization of the theory stays the same one gets to see a gross phenomenology that is very similar to the Fermi-liquid but in doing so the logic behind certain numbers may change radically: a high superconducting T_c becomes easy.

This story illustrates actually a key alluding to the uneasy relationship of mankind to science: we better regard ourselves as a biological species that should be designated as "religious". I encountered an interesting idea from Robbert Dijkgraaf in the context of a journalist who interviewed a couple of us for a feature in a dutch magazine dealing with the theme. Robbert argued that in order to survive as beasts on the savanna preconceptions associated with the nature of reality better be hard wired: one better starts to run immediately hearing the growl of a big cat, when one first takes the time for extensive theoretical considerations regarding the origin of these sounds one will surely be eaten.

Our human existence revolves around internalizing such dogma's, including matters such as the medieval certainty that the earth is in the middle. Only rather recently the Vatican admitted that Galilei guessed the right answer! But one encounters the same attitude of religious blindness in the daily practice of the research in a relatively humble subject such as superconductivity at a high temperature. BCS is presented to physics students in the same way as the genesis to young catholics. The (predictable) bottom line is that the "quantum critical" heresy is *completely* ignored! Our papers attracted very few citations. Even more painfully, I referred to cluster DMFT in Section (IV E): this is actually in the hands of a large community where the obligatory stuffs get cited thousands of times. In fact, under the supervision of the father of the cluster version (the late Mark Jarrell) direct evidence for relevant quantum critical BCS was reported in 2011 [55]. This paper is only cited ~ 50 times!

A recommendation in such circumstances is to find a way to mobilize experimentalists: as in normal life, these pragmatists are apt to defeat dogma's by tinkering in the lab. But with hardware comes hardship. We pointed out that the *dynamical* pair susceptibility can be measured employing a 1970's design resting on second order Josephson coupling [54]. A number of quantum transport specialists had a closer look but it all failed because of insurmountable material related difficulties.

4. The holographic Leiden-MIT fermions and the UV dependence.

It is just another fact of life in the laboratory that it is for various practical reasons difficult, if not impossible to measure *dynamical* response functions over a large kinematical range. We just encountered the impossibility to measure the pair susceptibility. Before that we encountered the "primitive" charge susceptibility – away from $q = 0$ where it is readily available by optics only very recently machinery became available to measure it with a sufficient energy resolution (Section IX B 3). The dynamical magnetic responses can be probed by inelastic neutron scattering but this is impeded by a signal to noise ratio that is so bad that the signal cannot be detected in the cuprate strange metal.

In fact, the only dynamical information that is readily available with very high resolution pertains to the *fermion* operators: the momentum space angular resolved photoemisson (ARPES) and its real space counterpart Scanning Tunneling Spectroscopy (STS) (see Section IX C). The next big splash following immediately after the holographic superconductivity was the discovery of the "Leiden-MIT holographic fermions". I was myself directly involved. This started in 2007 when I got on speaking terms with Koenraad Schalm (back then in Amsterdam) and I fired to him the question "any chance that you know how to compute fermion propagators holographically". Koenraad knew how to do this – it got delayed significantly by a bug in the code but eventually we nailed it with help of the graduate student Mihailo Cubrovic. Subsequently we managed to land the paper in Science [13]: the first ever string theory work that got published in this prestigious journal.

However, in the mean time greatly competent competitors had gotten the same idea: Hong Liu, John McGreevy and a number of brilliant students at MIT. They managed to take apart out numerical results to actually explain how it all worked in terms of bulk language. I already alluded to their penetrating insights in Section (VI C) identifying the $z \to \infty$ local quantum criticality as encoded in the throat geometry of the RN black hole. They explained also how this landed in the structure of the fermion two point functions in the boundary.

Although this work had the effect that I got pulled into the holographic effort, in the era that followed it was clarified that these fermions are an instance where the IR phenomenology is actually *critically dependent on the large N UV.* This was quite deceptive: initially it looked like that the holographic outcomes were remarkable look-alikes of the experimental results. But it became clear later that this was a deception (see also Section IX C).

ARPES is a close proxy to the measurement of the probability to remove an electron from the system at a particular energy and single electron momentum. STS measures a proxy to the probability to add and remove an electron, but now locally in space ($\mathbf{r} = \mathbf{r}'$). Both techniques are not ideal measurements, one has to be aware of various caveats but by and large the experimentalists have a fair understanding what these are. These therefore yield direct information on the central pillar of the classic diagrammatic perturbation theory of "particle physics" applied to electron systems: the one electron Green's function,

$$iG_{\alpha,\beta}(\mathbf{r}t, \mathbf{r}'t') = \langle \Psi_0^N | \hat{T} \left[\psi_\alpha(\mathbf{r}t) \psi_\beta^\dagger(\mathbf{r}'t') \right] | \Psi_0^N \rangle \tag{59}$$

specializing to zero temperature ($|\Psi_0^N\rangle$ is the ground state) while $\hat{T}$ is the time ordering operator and $\psi_\alpha^\dagger(\mathbf{r}t)$ creates an electron at $\mathbf{r}t$ with spin α. STS yields the local in space spectral function while ARPES delivers the momentum space spectral function for electron removal, associated with Eq. (59) in the frequency domain.

Long before these experimental techniques were developed, the quantum field theorists who developed the diagrammatic theory in the 1950's appointed this object as organizing principle: the "double lines" in the standard diagrams. But this rests on converging perturbation theory around the free fixed point: it only works departing from a SRE product "classical vacuum".

I presume the reader is familiar – it works the same way in the high energy context as in condensed matter, the main difference being that in the latter case one aims at the finite density Fermi gas. One departs from the non-interacting system, with the single particle electron states given by the quantum mechanical band structure $H_0 = \sum_{\mathbf{k},n,\sigma} \varepsilon_{\mathbf{k},n} c^\dagger_{\mathbf{k},n,\sigma} c_{\mathbf{k},n,\sigma}$ where n is a band label that I will suppress in the remainder to save writing indices. One the switches on the interactions $\sim H_1$. Under the condition that the perturbation theory is *converging* the effects of the interactions can be lumped together in the *self-energy* Σ,

$$G(\mathbf{k}, \omega) = \frac{1}{\omega - \varepsilon_{\mathbf{k}} + \mu - \Sigma(\mathbf{k}, \omega)} \tag{60}$$

The imaginary part of which is the spectral function that is measured experimentally. The existence of the quasiparticles is signalled by the presence of "poles". The self-energy is a complex function, $\Sigma = \Sigma' + i\Sigma$" and Σ" encodes for the inverse life time of the quasiparticle – the KK consistent Σ' will shift around the energy of the quasiparticle encoding for the generic mass enhancement of the quasiparticle relative to the bare particle. In the spectral function one will find a Lorentzian peak at this QP energy with a width $\sim \Sigma$", the "particle pole".

When Σ" decreases sufficiently fast upon approaching zero energy the quasiparticle is underdamped ("it exists") becoming infinitely long lived at the Fermi-surface. Departing from this fixed point one can get away with second order perturbation theory at low energy.

The excited electron/hole decays in the Lindhard continuum thereby picking up the power law behaviour. Accordingly, one is dealing with a near ideal gas like situation where the life time of the electron is set by the collision rate with other electrons, which is easily computed to be $\Sigma"(\omega, T) \sim g^2\left((\hbar\omega)^2 + (2\pi k_B T)^2\right)$. The dispersion $\varepsilon_k \sim v_f k$ and it follows self consistently that the interacting electron system renormalizes in the free quasiparticle gas.

Anticipating on the discussion regarding the experimental landscape in cuprates (Section IX), the evidence coming from STS and ARPES for the existence of such quasiparticles *deep in the superconducting state* is overwhelming. These appear to closely follow the expected behaviour for the Bogoliubov fermions of the BCS theory: $\gamma_k^\dagger \sim u_k c_k^\dagger + v_k c_{-k}$. Intriguingly, these spring into existence when the superconducting order develops. However, the question is whether these also exists in whatever incarnation in the strange metal above T_c.

Also in the normal state, pending the direction of momentum one still finds peaks moving as function of momentum (the"nodal fermions") and the habit is widespread to jump to the conclusion that there are still quasiparticles, Fermi surfaces and all of that. But this is a tricky affair. For instance, the CFT Fermions at zero density (Fig. 4) do exhibit peaks in the spectral function that disperse as function of momentum. The "unparticleness" is however encoded in the energy dependence of the line shape, branch cut propagators have an analytical structure that cannot be written in the perturbative self energy form Eq. (60). Such lineshapes are for practical reasons difficult to capture in experiment and I will discuss very recent progress demonstrating the unparticle nature of the strange metal in Section (IX C).

The Leiden-MIT fermions seemed to get close initially but it became later clear that this is actually a large N pathology. What is going on? This early work departed from the RN strange metal. According to the dictionary fermions are inherently quantum mechanical also in the bulk and one finds out that the two point fermion propagators in the boundary dualize in Dirac *quantum mechanical waves* "falling" to the black hole in the bulk. The MIT group reconstructed how this works in the bulk in terms of the "matching construction". For the Dirac fermions propagating in the bulk one can capture the effects of the change in geometry upon entering the RN throat in terms of an effective flat space Schroedinger equation: the "geometrical domain wall" signalling this geometry change at radial coordinate $\sim \mu$ translates in a large potential barrier felt by the fermion. This acts like a box "clamping" the Dirac waves at the boundary and the outer side of the barrier producing standing waves, corresponding with a tower of fermion states in the boundary. These can however tunnel through the barrier, landing in the near horizon regime where these fall through the horizon: the rate by which this happens translates into the life time of the quasiparticles in the boundary.

The bottom line is that the holographic fermion propagators reproduce the self-energy form Eq. (60). The $\varepsilon_{\mathbf{k}}$ are associated with the radial "standing Dirac waves" while the self energy is set by the tunneling to the near-horizon geometry. The action is at the holographic equivalent of the (large) Fermi-momentum and this probes the geometry at length smaller than the local length I discussed towards the end of Section (VI E 2). The deep infrared fermion propagator has here a similar odd form, depending on the fermion wavevector: $\Sigma(k,\omega) \sim \omega^{2\nu_k - 1}$ where $\nu_k \sim \sqrt{(1/\xi)^2 + k^2}$ where $\xi \sim 1/\mu$ is the local length.

The picture that emerges is that quasiparticles do exist but by second order perturbation theory (the tunneling in the bulk) they do encounter a "quantum critical heat bath" (the near horizon strange metal degrees of freedom) in which they eventually decay. This is in tune with the old "marginal Fermi-liquid" phenomenology, the difference being that the anomalous dimension governing the self-energy is now a free parameter, set by $k = k_F$. But for $\omega \to 0$ $\Sigma" \to 0$ and one can identify a Fermi surface – the big deal of our Leiden numerical work. The tricky part is that the exponent $\alpha = 2\nu_{k_F} - 1$ can vary all the way from 2 (the Fermi-liquid value) to smaller than 1. The case $\alpha = 1$ is the marginal case: by measuring the width of the peaks in momentum space as function of energy one sees such a damping in the ARPES (nodal fermions) of optimally doped cuprates, the stronghold of the MFL view. However, α can be less than one and then the self energy is more IR relevant than the dispersion: in the scaling limit $G \to 1/\Sigma$ and one is dealing with a "naive" emergent branchcut in the deep IR.

In fact, by tracking the momentum width behaviour as function of doping such a "running" α was extracted on formation, with $\alpha < 1$ in the underdoped regime. For those who pay attention, this is precisely the phenomenon happening in the large N limit of SYK, claimed to prove all kinds of holographic relations in $0 + 1$ dimensions. I am actually quite sceptical – precisely this feature is well understood to be highly pathological in controlled holography.

Hence, this looked early on quite promising. But I remember well sitting in on a talk by the string theory celebrity Joe Polchinski, explaining a paper he had written with Faulkner [56]. Rarely ever did I encounter such a disappointment dealing with theories of physics.

Joe explained that the above phenomena can actually be understood directly in the boundary field theory. But the big deal is that these are *critically dependent* on the fact that one is dealing with a supersymmetric Yang-Mills theory *in the large N limit*! This is just UV sensitive special effect of the large N limit and from the way it springs in existence one can directly conclude that this has nothing to say about the way that quantum supreme metals formed from electrons in solids behave.

The large N affair works is as follows. Given that one is dealing with a non-Abelian gauge theory one may not be surprised that even in the large N versions one is still encountering the generic physics of such theories: the vacuum may be *confining or deconfining*. All the "quantum supreme" powerlaw stuff is associated with the deconfining regime. It is well understood how confinement works in holography: in essence, the geometry of the deep interior just vanishes being replaced by "walls", geometrical structure reflecting the waves emanating from the boundary. This is the bulk "box" I referred to in the above and the various standing waves that form along the radial direction dualize in towers of states in the boundary corresponding with the gauge singlet "mesons" of the confining phase.

In fact, in this particular setting the fermions are the supersymmetric partners of the gauge degrees of freedom in the adjoint: in top-down set ups these can be for instance be identified with the SUSY partners of gravitons. The big deal is that in the large N limit the interactions between the gauge singlets ("pion exchange") is completely suppressed and the (s)mesons live infinitely long. This is the origin of the free limit in holography.

But the real bad news is still to come. As Joe explained, the Leiden-MIT fermions are nothing else than "mesonic resonances" that are developing in the deconfining state upon approaching the confinement transition tuned by varying the zero density fermion anomalous dimension to the unitary (free CFT) limit. But the nature of these resonances is determined by a large N pathology. One can demonstrate that all vertex corrections are $1/N$ suppressed and only under this condition the simple affair of quasiparticles decaying in a heat bath by second order perturbation theory makes sense! This is actually the same short cut taken in MFL – we know that this is a-priori unreasonable, it is just an intuitive shortcut. It is controlled in holography, unfortunately, by a small parameter that has no meaning in condensed matter ($1/N$).

This story continuous with the construction of the dual of the Fermi-liquid that followed soon thereafter: the "electron star" (see Chapter 11 in [9]). It may be already obvious that deep in the confining regime it is very simple construct a real Fermi-liquid. One is just dealing with the towers of non-interacting fermionic mesons. In the boundary one can just fill up these states using the Pauli principle to find an impeccable Fermi-gas. The more subtle question is, how is this encoded in the bulk? The fermions are $1/N$ suppressed and therefore individual fermions do not contribute to thermodynamics, let alone that collective (density etc) responses have anything to do with fermion loops: this is special effect of the Fermi-liquid. But a macroscopic assembly of fermions in the bulk can take over the ground state.

In the special, unphysical limit that the fermion charge becomes infinitesimal the bulk becomes tractable: upon lowering the fermion mass in the bulk, at a critical point the RN black hole "uncollapses" in the "electron star". This is nothing else than the Tolman-Oppenheimer-Volkoff solution for the neutron star, resting on the Thomas-Fermi equation of state for the Fermi gas, incorporating the electrical charge. This describes indeed the free Fermi-gas in the boundary. In fact, in this limit the electron star takes over from the black hole when the zero density fermion scaling dimensions become such that the Leiden-MIT probe limit indicates that Fermi surfaces start to form.

At the least this provides proof of principle that the holographic strange metals know about the fermion signs since these can be the birth place of Fermi-liquids. But even within the confines of the large N theory the precise workings of these holographic Fermi-liquids is far from settled. Upon increasing the fermion charge one finds out that one has to deal with a *quantized* electron star. It turns out to be very difficult for a variety of gravitational difficulties one encounters dealing which such highly quantal matter sources. This is presently still an open problem.

The take home message is that one should be at all occasions acutely aware that the strong emergence physics suggested by holography may be polluted by "UV dependence". Seemingly general traits of the IR theory may still depend in a critical way on the oddities of the UV theories of the string theorists. The good news is that the motives that spoil the fermions are tied to $1/N$ suppressed features. None of these seem to play a role in the leading order (in N) collective sector that delivers the charge excitations, transport, pair susceptibilites and so forth. Still, holography may be a worthwhile guide book telling us how to navigate our minds out of the quasiparticle tunnel vision. But this navigation tool can not be trusted, at unexpected instances it may be quite unreliable.

This terminates the first part of the discussion of the physics of quantum supreme states of matter according to holography. What remains to be done is the presentation of holographic transport theory, the most highly developed part of this agenda. But to get this in proper perspective the last technical hurdle has to be overcome: spoiling the homogeneity and isotropy of the space manifold. This is presently still a frontier where much is yet to be sorted out.

VII. BREAKING THE TRANSLATIONAL SYMMETRY AND HOLOGRAPHIC TRANSPORT THEORY.

A. The basics of transport theory: conservation laws.

The detailed explanation of the way that the transport of charge and heat works in normal metals is among the great success stories of solid state physics. The Boltzmann kinetic theory departing from the Fermi-liquid was cooked to perfection in the 1950's turning into a semi-quantitative framework explaining in detail how various transport behave pending the specifics of the electron system. Generation after generation got trained in this art and it seems that in the course of time a certain blindness developed with regard to distinguishing the generalities that govern any form of transports and the specifics of the gaseous physics in its most extreme form underlying Fermi-liquid transport.

Invariably, the macroscopic transport in metals is of the "Drude" kind. A student taking a bachelor course in solid state physics may have the impression that Drude transport is uniquely associated with the weakly interacting Fermi-gas. It does happen that I hear condensed matter physics professors claim that a Drude transport implies that one is dealing with a Fermi-liquid! It is actually the case that Paul Drude formulated his theory in 1900, before Planck identified his constant. He departed from a dilute classical plasma, resting on the fact that with kinetic gas theory one can keep track of all the details. But the outcome is actually completely generic: it is controlled by *conservation laws*, protecting the "hydrodynamical soft modes". The theory is generic, it applies to literally *anything*. Only the *numbers* are pending the specifics of the microscopic nature of the electron system.

This is very simple but the intervention of holography was in this regard for me personally quite beneficial in helping me to think outside the "quasiparticle box". Transport in holographic systems is entirely detached from the usual weakly interacting quasiparticles but it was at least psychologically an eye opener to see that it eventually boils down to the same Drude response as in the Fermi-gas of the textbooks. At the same time I got quite alert regarding widespread habits that had developed in the CM community in the course of time, such as the rather cavalier way to completely ignore the "transport vertices" in computing the current response functions entirely in terms of mere fermion-loops, referring to "local models". This is for instance the community standard in e.g. the use of dynamical mean field theory: frankly, I am unconvinced whether any of it relates to transport in the laboratory.

Let me present here a very elementary overview of the principles controlling transport before I turn to the specifics of the holographic metals.

1. Drude transport: generic at finite density.

Transport means that stuff is moving from A to B in a finite amount of time. One meets immediately the first Noether charge identified by mankind: when space is *homogeneous*, it looks everywhere the same, the total momentum of the stuff is *conserved*. It is Galilean invariance: your velocity is not changing when you fly through empty outer space.

The current is actually about the transport of mass, assuming for simplicity that the stuff has a finite rest mass. The most natural force to accelerate such stuff is gravity. But electrons are very light and the gravitational force is nearly vanishingly small as compared to the electromagnetic forces that can be exerted on the charged electrons. In X-ray tubes, synchrotrons and so forth one let electrons accelerate in the vacuum in an electrical field and everybody knows the outcome. Assume a system of electrons at a density n and one can relate the total charge current $\vec{J}$ to total momentum $\vec{P}$ by,

$$\vec{J} = ne\vec{v} = \frac{ne}{m_e}\vec{P} \tag{61}$$

However, when electrons move in solids with thermal energies they experience a space where the translational symmetry is weakly broken – why this is the case requires microscopy. This means that their total momentum acquires a finite lifetime, τ_P. This is all one needs to know to write the EOM for total momentum,

$$\frac{d\vec{P}}{dt} + \frac{1}{\tau_P}\vec{P} = e\vec{E} \tag{62}$$

where $\vec{E}$ is the applied electrical field. Assume this source to oscillate in time like $\vec{E}(t) = \vec{E}(\omega)e^{i\omega t}$ and look for a response $\vec{P}(t) = \vec{P}(\omega)(t)e^{i\omega t}$ and it follows immediately that $(i\omega - 1/\tau_P)\vec{P}(\omega) = eE(\omega)$. It follows for the AC (optical) conductivity $\sigma(\omega)$,

$$\begin{aligned} \vec{J}(\omega) &= \frac{ne}{m_e}\vec{P} = \sigma(\omega)\vec{E}(\omega) \\ \sigma(\omega) &= \frac{\mathcal{D}_D}{\frac{1}{\tau_P} - i\omega} \end{aligned} \tag{63}$$

Where $\mathcal{D}_D$ is the general quantity "Drude weight"; in this specific example it is coincident with the plasma frequency squared of the simple electron plasma $\mathcal{D}_D = \omega_p^2 = ne^2/m$ but in general this "screening power" of the electron system may involve microscopic specifics. The optical conductivity is a complex function $\sigma(\omega) = \sigma_1(\omega) + i\sigma_2(\omega)$, where the real part ($\sigma_1$) represents the dissipative part of the response. This is the "half Lorentzian" centred at zero frequency with a width $1/\tau_P$: $\sigma_1(\omega) = \mathcal{D}_D \frac{1/\tau_P}{(1/\tau_P)^2+\omega^2}$.

Taking now the DC limit ($\omega = 0$) it follows immediately for the DC resistivity that $\rho = 1/\sigma_1(\omega = 0) = 1/\mathcal{D}_D \times 1/\tau_P$, the familiar text book expression ($\mathcal{D}_D$ and τ_P are typically expressed in the specific dimensions of the Fermi gas). By practitioners of DC transport properties it is a habit to assume that one is dealing with Drude transport. But the only way to find out whether this is the case is by measuring the optical conductivity: the outcome Eq. (63) represents an analytical function that is uniquely related via Eq. (62) to the existence of a momentum that is sufficiently long lived to conclude that it actually controls the transport.

This simple ploy can be further formalized using the Mori-Zwanzig memory-function formalism. One can look this up, see e.g. [10] that is quite focussed on transport. The take home message is that it revolves entirely around modes that are to zero-th order protected by conservation laws, bringing into account the "weak" violation of the conservation law using perturbation theory. It highlights the difference with other forms of perturbation theory in having the hydrodynamical soft modes in the drivers seat.

One important generalization is that the relaxation rate may dependent on time itself. In the above we assumed that the origin of the momentum relaxation is static: $1/\tau_P$ is assigned to "elastic scattering". Using the memory matrix formalism it is easy to demonstrate that the momentum relaxation can be captured by an "optical self-energy" $\hat{M}$,

$$\sigma(\omega) = i \frac{\mathcal{D}_D}{\hat{M}(\omega) + \omega} \tag{64}$$

One recovers the "simple Drude" by asserting that $\hat{M}(\omega) = i/\tau_P$. However, this KK consistent quantity is in general frequency and temperature dependent. For instance I will explain soon that due to the presence of a periodic lattice (Umklapp scattering) $\hat{M}$ becomes in a Fermi-liquid $\mathrm{Im}(\hat{M}) \sim (\hbar\omega)^2 + (k_B T)^2$, it behaves as the quasiparticle collision rate.

A final general wisdom is associated with the fact that the optical conductivity and the charge susceptibility/dielectric function are tied together by the continuity equation. According to the Kubo formalism the conductivity and charge susceptibility are associated with the current-current and density-density correlation functions, respectively. But currents and densities are tied together by the continuity equation that is expressing charge (number) conservation, $\partial_t \rho + \vec{\nabla} \cdot \vec{\rho} = 0$. Accordingly, one can relate the response functions by,

$$\begin{aligned} \sigma(\omega, \mathbf{q}) &= i \frac{\omega}{q^2} \Pi(\omega, \mathbf{q}) \\ \chi_\rho(\omega, \mathbf{q}) &= \frac{\Pi(\omega, \mathbf{q})}{1 - V_q \Pi(\omega, \mathbf{q})} \\ \varepsilon(\omega, \mathbf{q}) &= 1 - V_q \Pi(\omega, \mathbf{q}) \end{aligned} \tag{65}$$

Where $V_q \sim 1/q^2$ brings into account that the zero sound that resides in Π gets promoted to the plasmon. The infinitely long lived sound at $q \to 0$ in the Galilean continuum translates in the delta function peak at $\omega = 0$ that one finds in his limit in σ_1. Different from χ_ρ that can be measured over a large momentum range by EELS, because of the fact that the light velocity is much larger than typical material velocities one can with optical means only measure $\sigma(\omega, q = 0)$. In fact the (longitudinal) conductivity shows the zero sound mode as a resonance at finite momentum as well as the imprint of the (generalized) Lindhard continuum.

In summary, the take home message is that this Drude response is actually completely generic for any system at a finite density characterized by a "weak" breaking of translational symmetry. On this level it has no relation whatsoever to the issue whether one has the extremely "gaseous" excitations of the Fermi-liquid or either with the densely entangled unparticle "soup" of the holographic metals. The transport in the latter is at energies compared to the chemical potential as much dominated by a Drude peak as it is in a simple metal like copper. However, there is still a difference rooted in a big difference in the UV between electrons in solids and the holographic CFT's. Electrons have a large rest mass (pair production threshold $\sim$ 1 MeV) while the CFT's are about an ultra-relativistic UV associated with zero rest mass.

B. Charge conjugation symmetry and the "incoherent" conductivity.

Is there any way for a physical system to evade a Drude response while the conductivity is finite? There is just one symmetry circumstance allowing this: *charge conjugation symmetry.* This is again very simple to understand. Imagine a system composed of a reservoir of positive- and negative charges that can move freely, while these reservoirs carry precisely the same net charge. Apply an electrical field: the $+$ and $-$ charges move in precisely opposite direction and an electrical current is running. However, since these reservoirs are otherwise identical the *center of mass of the combined system is not moving.* Hence, the electrical current $\vec{J}$ *decouples completely from total momentum* $\vec{P}$.

What to expect for the optical conductivity under such circumstances? Let us consider the case of massless, non interacting Dirac fermions. An example in condensed matter physics is graphene. Graphene is of course famous for having such "Dirac cones" in the band structure. But there is an issue with the interactions. Although much of the gigantic graphene engineering effort assumes that one can get away with ignoring interactions, departing from the (poorly screened) Coulomb interaction one can show that the interactions are marginally irrelevant [57]. The IR fixed point is free but interactions switch on rapidly upon raising temperature or energy. But this will not matter for the electrical conductivity as you will see in a moment.

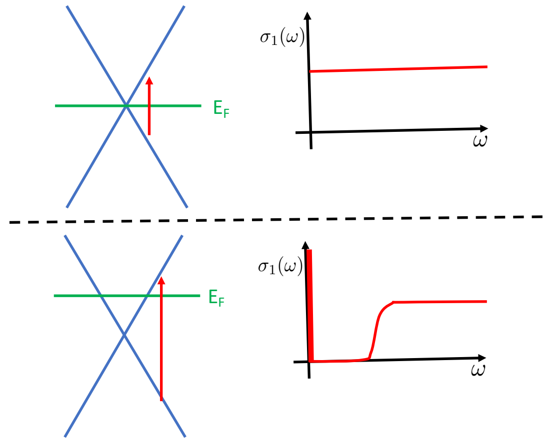

FIG. 12. The optical conductivity dealing with free Dirac fermions. At zero density (upper panel) the optical conductivity is due to "interband transitions" (red arrow) giving rise to an energy independent conductivity in two space dimensions. At a finite density (lower panel) these are exponentially suppressed at low energy due to Pauli blocking and this missing spectral weight lands in the Drude peak at zero frequency.

1. The engineering scaling of the CFT electrical conductivity.

The conductivity is now determined by the particle-hole excitations that can be created at a given energy, see Fig. (12). In a condensed matter language these correspond with interband transitions that are organized in a particular fashion. It is easy to count this out with the result that at zero temperature $\sigma(\omega) \sim \omega^{(d-2)/z}$. For Dirac fermions $z = 1$ but this includes for instance the case that two parabolic bands touch at the charge neutrality point as in bilayer graphene where $z = 2$. It follows immediately that in two space dimensions (as in Graphene) the optical conductivity is independent of frequency, a well known fact that is confirmed experimentally (Fig. 12).

But imagine now that we turn into it into a strongly interacting CFT such that the free Dirac fermions will turn into the branch cuts, what happens with the electrical conductivity? Yet again symmetry exerts control: next to energy conservation we know that the electrical charge is locally conserved, "protected by gauge-invariance". This implies that the current operators cannot acquire anomalous dimensions and the implication is that the engineering scaling of the free limit is actually universal. The ramification is that the optical conductivity should obey the following scaling relation [58] ,

$$\sigma_{\mu=0}(\omega, T) = \frac{Q^2}{\hbar} T^{(d-2)/z} \, \Sigma(\frac{\hbar\omega}{k_B T}) \tag{66}$$

where Q is the electrical charge. The energy/temperature cross over function $\Sigma(x)$ has to be of such a form that for $x << 1$ $\sigma \sim T^{(d-2)/z}$ while for large x $\sigma \sim \omega^{(d-2)/z}$: only the prefactors will be different in both limits . The take home message is that the conductivity acquires in $d = 2$ this universal frequency independent form in the presence of charge conjugation symmetry tied to zero density. This is surely confirmed by holography; in the plain vanilla "Maldacena CFT" one finds that the prefactors I just mentioned are the same due to a bulk electromagnetic duality [59].

2. Thermal transport and the violation of the Wiedemann-Franz law.

But this has striking consequences dealing with *thermal* transport. The electrical field is special in the regard that it pulls the positively- and negatively charged reservoirs in opposite directions. But imagine that we could source these with e.g. gravity: both reservoirs would move in the same direction resulting in mass transport implying that total momentum takes control: this would again result in a Drude type transport. This can be achieved by applying thermal gradients. The positive and negative "particles" carry the same amount of entropy and a heat current will start to flow "overlapping" (in the memory function language) with total momentum.

Given a Drude conductor, the way that thermal- and electrical conduction hangs together is canonical. This is generically governed in linear response by the following tensor of transport coefficients. In the presence of an electrical field E and temperature gradient ∇T an electrical (J) and heat (Q) current will start to flow according to,

$$\begin{pmatrix} J \\ Q \end{pmatrix} = \begin{pmatrix} \sigma & \alpha T \\ \alpha T & \bar{\kappa} T \end{pmatrix} \begin{pmatrix} E \\ -\nabla T/T \end{pmatrix} \tag{67}$$

where $\bar{\kappa}$ and α are the thermal conductivity and thermo-electric conductivity that is behind the thermopower. To illustrate the workings of Drude transport a bit further, in a relativistic system one obtains the following expressions for the various optical conductivities [60]. Define the energy density an pressure as $\mathcal{E} + P$ and the charge density and chemical potential as Q and μ, the entropy density s while the momentum relaxation rate is Γ,

$$
\begin{aligned}
\sigma(\omega) &= \frac{Q^2}{\mathcal{E}+P}\frac{1}{\Gamma - i\omega} + \sigma_Q \\
\alpha(\omega) &= \frac{Qs}{\mathcal{E}+P}\frac{1}{\Gamma - i\omega} - \frac{\mu}{T}\sigma_Q \\
\bar{\kappa}(\omega) &= \frac{s^2 T}{\mathcal{E}+P}\frac{1}{\Gamma - i\omega} + \frac{\mu^2}{T}
\end{aligned}
\qquad (68)
$$

One discerns the Drude form for σ: substitute $Q^2 \rightarrow (ne)^2$ while for a system characterized by a large rest mass $\mathcal{E} + P \rightarrow m_e n$ and the Drude weight becomes the familiar $\omega_p^2 = ne^2/m_e$. σ_Q is special for zero rest mass systems – I will come back to this later and you can ignore it for the time being.

One discerns that all these coefficient share the same momentum-relaxation factor $1/(\Gamma - i\omega)$, the differences are entirely in the weights. One discerns that when all these quantities are available one can actually find out from only DC transport data whether one is dealing with a Drude transport. All one needs in addition are the thermodynamic quantities charge density and entropy that can be determined independently.

This includes the Wiedemann-Franz(WF) ratio $L = \kappa/(\sigma T) = (s/Q)^2$, this just counts the entropy carried by the current per unit of charge. Given a Sommerfeld entropy and a temperature independent carrier density this becomes just a number – in the Fermi gas this corresponds with the Lorentz number $L = \pi^2 k_B^2/3e^2$.

But let us know consider zero density, as controlled by charge conjugation symmetry. The charge density $Q = 0$ while the entropy density is finite at finite temperature. The Drude part in σ is vanishing: what remains is the zero density "incoherent" component, Eq. (66): this is referred to in Eq. (68) as σ_Q where "Q" refers rather awkwardly to "quantum critical". This is inspired by the branch cut form of Eq. (66) but in fact it only relies on the fact that the Drude is completely suppressed. There is of course scale invariance wired ineven in the free limit by the absence of a mass scale in the free dispersion. This naming echoes desires early on to somehow connect holography to zero density QCP's.

The bottom line is that at zero density the Drude thermal conductivity and the "branch cut" electrical conductivity completely loose the "Drude" synchrony, exhibited by a strongly temperature dependent WF ratio. This was observed to be indeed the case in graphene at charge neutrality [61].

3. The holographic "quantum critical second sector" at finite density.

I already alerted the reader that one should be always on the outlook for "UV dependence" in holography, the fact that the large N CFT at zero density may cause special effects also in the strong-emergence IR of the holographic finite density metals. I already explained the disastrous effect of large N in obscuring the physics of the fermions in Section (VI E 4). However, in this transport context there is yet another symmetry condition hard wired in the UV, being less threatening but yet again giving rise to a pathology when one wants to apply the holographic wisdoms to e.g. cuprate electrons. Electrons have a rest mass ~ 1 MeV, but holography can only be made to work for ultrarelativistic matter with vanishing rest mass: the zero density CFT's.

As I stressed over and over again, AdS/CMT is the story of what may happen when one departs from such a zero density, zero mass densely entangled stuff that is subsequently "doped" to finite density. The closest CM proxy would be graphene: when it would form a strongly interacting zero density critical state it could become a literal incarnation of a holographic strange metal at finite density. This does not seem to be the case: for good reasons finite density graphene appears to be a rather perfect Fermi-liquid.

Let us first find out what to expect for the optical conductivity when the *free* zero density Dirac electron system is put at a finite density. Shift up the chemical potential/Fermi energy in the Dirac cone (Fig. 12). Because of the Pauli blocking the "vertical" transitions giving rise to the frequency independent optical conductivity at zero density are completely suppressed. An absolute "hole" appears in the spectral weight, being exponentially suppressed at finite temperature for $\omega < E_F$, recovering above E_F. Given the f-sum rule one can immediately conclude that this weight will accumulate in the Drude peak forming a delta function peak at zero frequency in the homogeneous background.

This is the way it works with the fundamental QED response of real electrons. In order to discover the positrons one has to exceed the pair production threshold: at lower energy the density of electron-positron is suppressed exponentially. But this is a special trait of the non-interacting theory. What has holography to say about it?

The outcome is remarkable and a bit mysterious. Instead of the hard, exponential pair production gap the pair production spectrum "smears" in a powerlaw affair! The result is that in the deep IR a probed by DC transport two conducting fluids exist in *parallel* ("anti-Mathiessen"): on the one hand a Drude sector, living side-to-side with an incoherent sector that is invariably irrelevant towards the IR. This is the "σ_Q" in Eq. (68): $\sigma_Q(\omega = 0, T) \sim T^{\alpha}$, a branch cut submitting to ω/T scaling. As shown by Ref. [62], by taking particular combinations that can be computed of electric fields and temperature gradients one can separately source both fluids, thereby rendering their existence to be observable.

This is completely relying on holography – it is still a bit of a mystery of how to interpret the separate existence of these two fluids in the field theory language. The Drude part is easy, but what distinguishes the "incoherent sector" so that it has an existence by itself? So much is clear that it is somehow controlled by an *emergent* charge conjugation symmetry, a necessary condition to avoid the Drude logic.

This "two-sector" affair caused at least in my environment initially quite some confusing: is the incoherent pair-production affair a generic property of the densely entangled "generalized Fermi liquid" (e.g., Ref. [49])? A simple argument demonstrates that this cannot possibly be the case: it is a UV sensitive affair, and the existence of the second sector is critically dependent on the fact that the UV degrees of freedom (zero density CFT) are ultrarelativistic, characterized by the absence of rest mass.

The argument is relying yet again on symmetry. The two sectors emerge in homogeneous space: the Drude part is here the delta function at zero and the pair production sector appears in the optical conductivity as a spectrum of excitations that grows algebraically with energy. Consider now a system of UV degrees of freedom having a rest mass that is 9 orders of magnitude or so larger than the energy in the experiment while at the scale of this rest mass one is dealing with particle physics: the electrons in solids. Under these circumstances the momentum- and electrical currents have to overlap for the full 100%: at least at any finite temperature where one can use hydrodynamics *all* the spectral weight has to be in the Drude part. The fundamental electrons just behave like the band structure graphene electrons: pair production is exponentially suppressed, the density of positrons in an electron system at a temperature of a couple of Kelvins is of course completely vanishing.

Hence, the second sector is at least in the case of a homogeneous spatial background a special effect associated with the ultrarelativistic nature of the holographic UV. This causes more shortcomings when applied to the condensed matter electrons.Yet another ramification of the large rest mass of the electron is the decoupling of charge and spin. At best one has to account for the small spin-orbit coupling. But when the rest mass is vanishing one encounters the spin-orbital locking of the helical Dirac states and the spins disappear as separate decrease of freedom: in holography there is no room for the order of spins in the form of (anti) ferromagnetism.

The technical difficulty is that it is just not known how to incorporate finite rest mass in the zero density holographic set ups, a necessary condition to get a view on the effects of a finite mass UV. For the time being, all one can do is to just not trust anything that is rooted in the presence of the "second sector". As you will see, upon breaking the translational symmetry of the spatial manifold there are other ways that incoherent excitations may arise in the AC conductivity but also in these cases one should be aware of the possible "pollution" due to the massless UV.

Another aspect that I ignored discussing the holographic superconductivity in Section (VI E 3) is that deep in the holographic superconductor one finds an emergent deep interior scaling geometry. At zero temperature the "scalar hair" takes over completely from the black hole – this applies for instance to the Einstein-Maxwell-Scalar theory where one would find the RN extremal black hole when the scalar is switched-off. These in turn dualize in incoherent low energy excitations in the boundary. There is a claim that these can be classified [63] and pending the specifics of the set up it may even happen that this "rediscovers" the pristine zero density AdS geometry.

This is one of the mysteries of holography. As I explained in the beginning (Section (II B) spontaneous symmetry breaking is exquisitely associated with the 'classical" SRE product state vacua. Is it so that holography is telling us that part of the densely entangled strange metal is "grabbed" by this classicalness, leaving however behind a reconstructed densely entangled affair being responsible for the incoherent low energy excitations deep in the superconductor as well as other orderly states (the "mechanism" seems universal)? Or does it have dealings with "UV rubbish" (like the second sector) exclusively associated with the baroque large N etc. UV? The answer is at the time of writing these words in the dark.

C. Breaking translations in holography.

In so far holography was involved, up to this point we were dealing invariably with the way it works in the translationally invariant homogeneous space. Given the universality of Drude transport, anything characterized by broken charge conjugation symmetry has to behave like a perfect conductor given the conservation of total momentum implied by the homogeneity of the spatial manifold.

But electrons in solids do not live in such a space: these invariably encounter the lattice formed by the ions. This may be in the form of a perfectly periodic crystal structure, that may also be subjected to imperfections: the quenched disorder. We have learned to appreciate holography as a symmetry processing machinery: as it turns out, the (in)homogeneity of space is in this regard a *critical* symmetry condition dealing with equilibrium.

This is deeply rooted in the way that the Einstein theory in the bulk is stitched together. The first confession one encounters in any GR textbook is that the results that are highlighted are all exclusively tied to *highly symmetric* circumstances. This is presented with a great degree of mathematical discipline. The isometries of the problem translate in Killing vectors and when there are sufficiently many of them the mathematical abyss of the Einstein equations as a system of highly non-linear partial differential equations ("PDE's") can be reduced to an ordinary differential equation that can be solved on the backside of an envelope: e.g., the Schwarzschild solution, the FLRW cosmology and so forth. But if one breaks the homogeneity and isotropy of the spatial manifold hell breaks loose: one has somehow to "tame the PDE's".

The same applies to non-stationary conditions in general. Dynamical GR is not at all charted. A case in point one encounters in the fluid-gravity duality that I discussed in Section (V C). The non-stationary near horizon gravity is identified to be in precise dual relationship with the Navier-Stokes hydrodynamics in the boundary. But the best known of all dimensionless parameters is found in hydro: the Reynolds number R that is inversely proportional to the viscosity. Given the minimal viscosity $\eta/s \sim \hbar$ the viscosity can become quite small at low temperature and it is easy to get into a flow regime governed by large Reynolds number: but this implies that the fluid flow is in a *turbulent* regime. But given fluid-gravity duality this implies that the near horizon gravity is also exhibiting turbulent complexity!

The fluid-gravity specialists are still struggling getting this under control, although there is progress. I heard claims that during black hole mergers the conditions may be met for such turbulent horizons to play a role. Ironically this is actually challenging the good taste of the mathematical community. There is a Millennium prize by the Clay institute on Navier-Stokes, implicitily revolving around the turbulent regime. But we learn that Navier-Stokes is actually a tiny, very special part of GR. There is no Millennium prize for GR!

Yet again, in the present time the big GR effort is in first instance focussed on black hole mergers given the arrival of the gravitational wave detectors. Numerical GR solutions are an absolute requirement to interpret the signals, and also in this regard this heroic effort has been much helped by a good fate. It took roughly half a century of intense effort by a small expert community to get the numerical GR codes working, being just in time to interpret the first merger signals! In fact, these only work involving specific simplifying circumstances and a worldwide effort is presently evolving to further generalize this numerical RG.

As it turns out, as a spin-off of this effort efficient algorithms appeared that do not quite work for the dynamical problems while these are good enough for the stationary problems with low spatial symmetry associated with equilibrium holography. A few dispersed shots were launched some ten years ago to demonstrate proof of principle that this holographic numerical GR does work but not much happened since them. So much is clear that the breaking of translations interferes critically with holography. Given all the extra work to be done in the bulk it is not surprising that new holographic principle may arise being critical for the application to the electron systems in solids. Metaphorically it is like trying to explain the physics associated with the band structures of real solids having only insights into the quantum mechanics of particles in a box and the Pauli principle.

The reason that this did not trigger a big effort appears to be rather rooted in cultural factors. The string theory community has always been focussed on mathematical approaches and there is no computational tradition. Even the basic research culture is quite different. Computational physics is in a way much closer to experimental physics than to the mathematical culture of string theory. As in experiment one needs a small army to get the machines – the codes – working. When the codes start working these are like the experimental machines: in essence black boxes that produce data. One has then to systematically collect the data to then try to find out whether one can discern a phenomenological framework revealing the deeper meaning behind the data. This is quite different from the mathematical tradition where by solving the equations one obtains directly an overview of anything that can happen.

The bottom line is that when it became clear that a further exploration of AdS/CMT had to rely in numerics the string theory community at large shied away and went elsewhere. We felt responsible in Leiden and we have been investing in a professional computational holographic effort in the form of a program package written in efficient programming language (like C++, the industry standard among string theorist is mathematica!) running on supercomputers. The first results are obtained making above all clear that we have seen only the tip of the iceberg. I will report on some of it at the end of this chapter

Transport properties are at the centre of this "inhomogeneous numerical GR". But before we get there let us first step back to recollect the effects of the breaking of translations in the familiar Fermi-liquids in order to get the differences in sharp perspective.

1. Momentum relaxation in Fermi-liquids.

According to the universal Drude logic, in order to have a finite resistance the macroscopic current should have a relaxing total momentum which in turn requires a space lacking translational invariance. This pertains to literally anything – one could attach electrostatically charged cat furs to elephants that one then throws out a space ship in outer space. This flow of charged elephants represents perfect conduction. The issue is that the *numbers* in the Drude conductor do know that low temperature Fermi liquids do behave quite differently from charged elephants.

The big deal is the periodic ion lattice. I will largely ignore the impurities – quenched disrder appears to be a bit of an afterthought in e.g. the cuprate metals and the "Umklapp" is just more involved and interesting than weak disorder. The Fermi-liquid is in this regard different from any classical ("molecular") fluid in the regard that due to quantum mechanics the breaking of translations by a periodic lattice is *irrelevant at the IR fixed point.* At zero temperature a macroscopic Fermi-liquid living in a perfectly periodic potential forgets entirely the existence of such a potential turning into a perfect metal: the residual resistivity disappears in the absence of impurities.

This is rooted in the fact that the states near k_F that do the work are living in momentum space: these are delocalized in real space and this averages out the periodic potential. The thermally excited quasiparticles can only dissipate the total momentum at a quasiparticle collision where the lattice can absorb part of their centre of mass momentum – of course two particle momentum exchange does not affect the total momentum otherwise. Consider a potential $V = V_G \cos(\vec{G} \cdot \vec{R})$ where $\vec{G}$ is the Umklapp wave vector. The operator that dissipates momentum is of the form,

$$\hat{O} = \int (\Pi_{i=1}^{4} d^d p_i) c^\dagger_{p_1} c^\dagger_{p_2} c_{p_3} c_{p_4} \delta(p_1 + p_2 - p_3 - p_4 - G) \tag{69}$$

But to relax the total momentum this action at the large Umklapp vector should get coupled to the macroscopic current. This was elaborated by Lawrence and Wilkins in the 1970's [64] in terms of "Umklapp efficiency". The single particle continuum states near k_F acquire an admixture of the Umklapp copies $|k\rangle \rightarrow |k\rangle + \delta_{V_0}|k \pm G\rangle$ in the guise of weak potential scattering. In case that e.g. $G < 2k_F$ one finds this to be associated with the "efficiency" $\Delta_G \simeq \frac{\pi}{4} \frac{G}{k_F} \frac{V_G}{E_F}$. The rate by which momentum is absorbed is set by the two-quasiparticle collision rate, $1/\tau_c \simeq (k_B T)^2/(\hbar E_F)$. The momentum relaxation rate becomes then $\Gamma_G = \Delta_G/\tau_c \sim T^2$. This is the story behind the T^2 resistivity, dominating at low temperatures in conventional metals before the phonons take over the momentum relaxation.

One infers that $\Gamma_G \rightarrow 0$ because $\tau_c \rightarrow \infty$ when $T \rightarrow 0$: the IR irrelevancy of Umklapp and the effective Galilean invariance in the deep IR is therefore an emergent global symmetry. There is profundity in this simple story: in a classical fluid this cannot happen.

But we are not done yet. Dealing with ^{3}He one immerses the fluid in a truly translationally invariant background and I highlighted in Section (V C) that this behaves at a finite temperature as a Navier-Stokes fluid, characterized however by a very large viscosity $\eta_{FL} \simeq n E_F \times \tau_c$. The momentum exchange between the quasiparticles that is responsible for the viscosity takes a time τ_c becoming very long at low temperatures. The associated collision length $l_c \sim v_F \tau_c$ is easily of order of many microns at low temperatures. But it is extremely difficult to render crystals to be so perfect that they are devoid of any disorder on such large scales. Generically the *elastic* mean free path l_{mf} is much *smaller* than the collision length. At the "collision" with the impurity the individual quasiparticle dumps its momentum in the lattice and long before the system finds local equilibrium so that it can behave as a hydrodynamical fluid total momentum has already disappeared in the UV of the individual quasiparticles.

This is the underpinning of the standard kinetic transport theory that is entirely revolving around how individual quasiparticles loose their single particle momentum. Only very recently extremely clean conductors became available where it is claimed that $l_c < l_{mf}$ so that one can look for the signatures of hydrodynamical electron transport in mesoscopic transport devices, graphene being the prime example [65]. But one should be acutely aware that this extremely "gaseous" behaviors where quasiparticles can fly forever is extremely special for the Fermi-liquid. Despite the folklores imprinted by 80 years of Fermi-liquid success, why should any of this extreme behaviour survive in anything that is not a Fermi-liquid?

2. Minimal viscosity and the shear drag resistance in holographic fluids.

In our daily human world we encounter all the time the resistance due to fluids, be it that it limits the speed of cars or either that we need pumps to get water through a pipe. At least in the regime of low Reynolds number with its smooth flows this is due to *shear drag*. The breaking of space translations due to e.g. the walls of the pipe cause gradients in the flow and when the viscosity is finite this dissipates the fluid kinetic energy into heat.

To estimate this resistance one can rely on very simple dimensional analysis. The viscosity can be converted into the kinematic viscosity (or diffusivity) through $\nu = \frac{\eta}{\mathcal{E}+P} = \frac{\eta}{mn}$ in the relativistic and non-relativistic fluid, respectively. The kinematic viscosity has the dimension of diffusion, $[\nu] = \mathrm{m}^2/\mathrm{s}$. Introduce now the length associated with the distance where the breaking of translations becomes manifest, l_η: this can be e.g. the radius of the pipe. The momentum relaxation rate can now be estimated to be of order,

$$\Gamma_\eta \simeq \frac{\nu}{l_\eta^2} = \frac{\eta}{mnl_\eta^2} \tag{70}$$

Consider now any holographic fluid living in a system where translational invariance is broken by a low density of impurities such that the typical wave vectors $\sim G$ are quite small as compared to e.g. the chemical potential. I already explained in Section (V C) the triumphant minimal viscosity predicted for the finite temperature CFT. The argument revolved around a bulk universality: the viscosity is set by the absorption cross section of zero frequency gravitons by the black hole and this scales with the area of the horizon as does the entropy. Therefore $\eta/s = A_\eta \hbar/k_B$ where $A_\eta = 1/(4\pi)$. But this is completely universal, it also applies to the finite temperature (and even the extremal zero temperature) black holes that are dual to the finite density strange metals!

Plugging minimal viscosity in Eq. (70) and using relativistic units it follows for the momentum relaxation rate setting the Drude conduction,

$$\Gamma_{holo} \simeq \frac{A_\eta s}{(\mathcal{E} + P) l_\eta^2} \tag{71}$$

The momentum relaxation rate and thereby the temperature dependence of the resistivity is according to holography in this regime just proportional to the entropy! It is very simple but it should be a bit of a shock for practitioners of conventional metal transport theory.

In the context of cuprates it is quite suggestive, see Section (IX B 1). In the strange metal regime the entropy has been measured to be Sommerfeld, linear in T. We know experimentally that the Drude weight is temperature independent: it follows from Eq. (71) that therefore the resistivity should also be linear in temperature, the holy grail of the cuprate strange metal. In fact, we appear to know all the numbers except l_η. To get that $\Gamma \simeq K_B T/\hbar \; l_\eta$ has to be of order of a reasonable couple of nanometers [11].

As will become further substantiated underneath this "shear drag" appears to be a rather universal result in holography at least dealing with small wavevector components of the symmetry breaking potential where one can actually discern a long wavelength universality associated with the way that the breaking of space translations is encoded in the gravitational bulk. Momentum relaxation is associated with a *shear* stress exerted by the spatial inhomogeneities on the fluid flow. Imagine water flowing through a channel littered with a periodic array of obstacles: velocity gradients transversal to the flow will develop dissipating the flow given that the viscosity is finite.

You learned from e.g. the minimal viscosity mechanism that the spatial shear components of the boundary energy stress tensor are dual to *gravitons* in the bulk. In a homogeneous space gravitons are massless and zero mass in the bulk encodes for the conservation of total momentum. Now we encounter a simple wisdom that appears to have been overlooked completely in the history of GR. How does an Einstein universe looks like when the Energy stress on the r.h.s. of the Einstein equation is associated with matter that breaks translational invariance? In other words, what happens when the universe would be filled with a "cosmological" crystal?

The answer is [66]: this corresponds with the *Higgs phase* of gravity where the background geometry takes the role of the gauge fields and the crystal that of the matter. In a way it is the most intuitive way to understand Higgsing in general: the crystal forms a "preferred frame", a coordinate system that is imprinted in the geometry by gravitational backreaction. As in the usual Yang-Mills setting, the geometrical curvature can now only be accommodated in the form of "gravitational fluxoids" as the Abrikosov quantized magnetic fluxes in a superconductor.

At long wavelength this spatial frame fixing has as ramification that the graviton acquires a *Higgs mass*. Massive gravity has a history in cosmology. This was however plagued by inconsistencies related to the (implicit) assertion that also time translations (unitarity) should be broken which is an unphysical affair. But exclusively fixing the spatial frame is healthy – it is just "crystal gravity"[66]. One now discerns the universality in the workings of the dictionary. In the boundary one applies a periodic potential with a small wavevector G. This dualizes in the bulk to a "fixed spatial frame" Einstein geometry having the universal long wavelength (small G) ramification that the graviton acquires a mass. The mass of the graviton in turn dualizes into a finite life time for the macroscopic total momentum: the dictionary entry for the Γ in the Drude response Eq. (68) is identified.

Using an effective field theory ploy the graviton mass can be easily incorporated in the bulk – it just parametrizes the l_η of Eq. (71). Although we understood this less well at the time, in Ref. [67] we worked out an explicit holographic set up illustrating precisely the above mechanism. We departed from the "conformal to AdS2" strange metal, that I already advertised in Section (VI D) as the only loop hole that I know to reconcile $z \to \infty$ with a Sommerfeld entropy, $s \sim T$. We added the massive gravity term to the bulk action, to find a *perfectly linear resistivity* all the way up $k_B T \simeq \mu$. It is just explicit holographic proof of principle demonstrating that indeed the very simple Eq. (71) is all one needs to know.

The big picture is that perhaps nowhere else one discerns such a sharp contrast in the physics of conventional Fermi liquid metals and the predictions of holography for the densely entangled "quantum supreme" metals. In a Fermi-liquid one approaches at low temperature the extreme gaseous fluid behaviour more closely than anywhere else: the quasiparticles behave completely independently up to the collision length scale l_c becoming very large at low temperature. The ramification is that the momentum relaxation is entirely due to single particles scattering against the impurities. But the rapid equilibration of the holographic fluid as rooted in the Eigenstate thermalization of the densely entangle "unparticle soup" has the consequence that at microscopic scales the fluid behaves cooperatively as described by hydrodynamics. This is then in a way a simplifying circumstance since such a fluid behaves like a macroscopic "molecular" fluid like water flowing through a space with obstacles.

The difficulty is to distinguish these very different transport regimes in experiment. At distances $>> l_\eta$ in both cases one gets the simple diffusional Ohm's law behaviour. One therefore has to employ submicron transport devices to look for the signatures of hydrodynamical flows. For instance, when the ploy in the above would be realized the implication would be that the viscosity itself would become very small at low temperature since we just know from experiment that the Sommerfeld type entropy becomes very small as in a Fermi gas, $s \sim T/\mu$. It is then relatively easy to get into a flow regime characterized by large Reynolds numbers by e.g, injecting a current through a small constriction. Although the precise ramifications are not worked out one would expect to see signatures of this in the nano-transport machinery. But this is very hard to realize because of the difficulty to reliably "nano-engineer" chemically complicated substances like the copper oxides.

Yet another simple ramification is in the fact that the minimal viscosity implies that the viscosity is vanishing when the entropy is vanishing: asserting the absence of zero temperature entropy this implies that the viscosity is vanishing in the $T \rightarrow 0$ limit. The holographic strange metals turn into *perfect fluids* at zero temperature and perfect fluids are perfect conductors! In sharp contrast with Fermi-liquids there should not be a residual ($T = 0$) resistivity. This may relate to strange anomalies in cuprate transport that I will discuss when I turn to experiment.

Finally, all along there is the warning of UV sensitivity: is the minimal viscosity in the finite density systems indeed universal or somehow tied to e.g. the large N limit? There is a reason for concern [11]. I gave the simple dimensional analysis argument for the minimal viscosity in the zero density CFT in Section (V C). The dimension of viscosity is set by the free energy density times a characteristic momentum relaxation time; the former is purely entropic in a CFT $\sim sT$ while the time is just $\tau_\hbar \simeq \hbar/(k_B T)$ and it follows that $\eta/s \sim \hbar$. But in a finite density system the free energy at low temperature is dominated by the density, $f \sim \mu n$, and combining this with the Planckian time one gets $\eta \sim 1/T$, growing not as rapidly with decreasing temperature but still divergent in the zero temperature limit. Especially Hartnoll has been stressing this, arguing that the minimal viscosity may be an artefact of the large N limit. But this is controversial – yet again, an experimental verdict would be most useful in shedding light on what amounts deep inside to a deep question in quantum gravity.

D. The Umklapp-mechanisms according to inhomogeneous holography.

How to proceed from the long wavelength but otherwise unstructured momentum dissipation encoded by massive gravity? Other holographic ploys became fashionable that do somehow incorporate more structure while these are still resting on the simple spatially homogeneous bulk. These "linear axions" and "Q lattices" do break translations and dissipate momentum but in such a way that one can still compute matters with simple ordinary differential equations. I will ignore these here – these are motivated by the ease of computation while it is completely in the dark what kind of physics is described in the boundary. Whatever it is, it is surely not something that is in any obvious related to the conditions met in the electron systems.

As in the Fermi-liquids the interest is in the first place in the mechanism leading to momentum dissipation by the periodic lattice, the "Umklapp". This is a condition invariably encountered in any solid. Specifically for the copper oxides, these crystals are far from perfect but the degree of disorder appears to vary quite a bit between the different subfamilies. However, it appears that the typical strange metal properties are rather independent of the degree of quenched disorder. The basic Umklapp departing from the perfectly periodic lattice appears to be the big deal. But to get a handle on how this works one *has* to sacrifice the homogeneity and resort to the numerical RG.

As I already stressed, this is hard work and very little is known. The effort in Leiden just started to get on steam and we are nearly finished with a first such study of the most basic holographic system of the kind. We considered the RN strange metal in two space dimensions on a background formed by a square lattice single harmonic electrostatic potential,

$$V = V_0 \left(1 + A(\cos(G_x x) + \cos(G_y y))\right) \tag{72}$$

with a strength A and an Umklapp wavevector of magnitude G. As I explained in Section (VI) its special deep interior scaling geometry is pathological and it is presently nearly completely in the dark how the Umklapp works in other strange metals characterized by arbitrary z's and θ's. The RN affair is however already interesting and surprising, showing at the least that momentum dissipation by Umklapp is governed by an entirely different physics compared to the Fermi-liquid "T^2" case that I outlined in Section (69).

1. The $z \to \infty$ holographic "pseudopotential".

Yet again the big deal are the scaling properties of the strange metal. The RN metal is characterized by $z \to \infty$, a condition that is of interest since there appears to be direct evidence for such scaling in the cuprate strange metals. But what does this mean for the way that such a metal reacts to an external potential both in terms of screening properties and Umklapp momentum relaxation? The text books only tell us how this works in Fermi-liquids: I already alluded to the "Umklapp efficiency" while the response to an external periodic potential is also a canonical affair. The latter is in essence governed by Fermi-pressure. The linear response of the charge density to a potential with wavevector G is given by the real part of the charge susceptibility at zero frequency and it follows immediately from the Lindhard function that $\mathrm{Re}\chi_\rho(\omega = 0, G) = \partial n / \partial_u$, the compressibility of the Fermi gas modulo possible "$2k_F$" enhancements due to nesting coincidences. It is in essence spatial scale independent and it takes care that the charge density is smoothened out compared to a non-degenerate system.

However, the Fermi liquid is $\theta = d - 1, z = 1$. What to expect for $z \to \infty$? This refers to the local quantum criticality affair while $\tau_{\mathrm{cor}} = l_{\mathrm{cor}}^z$: although τ_{cor} is diverging for $z \to \infty$ it is not at all clear what is going on with the *spatial* organization of the system. The usual correlation length looses its meaning in a local quantum critical system.

Yet again, Hong Liu and coworkers [51] showed their muscles by discovering a quite surprising structure hard wired in the bulk geometry of the RN system. Perhaps the greatest wonder of GR is that causality structure, the way that cause leads to effect, is hardwired in the Lorentzian signature geometry itself. The grand master of this affair is Roger Penrose who devised the Penrose causality diagrams but also got rewarded a Nobel prize in 2020 for his "trapped surfaces", causal objects that are at the heart of the singularity theorems. The take home message is that causality structure may be used to extract universal principle even when explicit solutions are beyond reach. In the greater context of whether one can trust holography, when a particular boundary physics is rooted in such causality structure in the bulk one better takes it very serious as an affair having the best chance to reveal correct general principle ruling in the boundary.

Pristine AdS is already exhibiting strange causal attitudes: it is a GR classic that although the radial direction extends to infinity it takes only a finite time to travel from the boundary to the deep interior, a motive that is actually critical to get the correspondence to work. But let us now consider the extremal RN-AdS geometry. Hong Liu and coworkers discovered the following causal structure hard wired in this geometry. Take two points in the near-horizon geometry with a spatial separation l. Launch light rays at these two points and follow their nul-geodesics to the boundary, to find out that when these points are farther apart than a distance $\xi = \pi/(\sqrt{2}\mu)$ (in $d = 2$) the ensuing light cones arriving at the boundary will not overlap. This implies that one is dealing with patches of size ξ in the boundary theory that do not correlate with each other at distances larger than their size. In conventional units, $\xi \simeq 1/k_F$ and the universal prediction is that the system will react quite rigidly to a potential with $G > 1/\xi$ while it becomes very soft at larger length scales!

This causality structure turns out to be universal for *all scaling geometries characterized by* $z \to \infty$. The hyperscaling violation does not affect this causal structure at all – more reason to take it serious.

The ramifications for the boundary can be illustrated in a spectacular manner by reference to the geometrization of the RG flow. The electrical field in the bulk is dual to the charge density in the boundary, and by plotting the way that the spatial structure of this electrical field varies along the radial direction one gets an image of the RG flow of the density in the boundary theory. A typical result is shown in Fig.(13). In the right panel the situation is indicated for a harmonic potential with a $G > 1/\xi$: a charge modulation is present at the boundary but when one dives to the horizon one sees that it becomes very small. At low energy there is barely any response to the potential and the system is nearly homogeneous. However, dealing with a $G < 1/\xi$ (left panel) the UV response (density in the boundary) is not only much larger, but upon descending along the radial direction the modulation stays very large all the way to the horizon. The periodic potential hits hard also in the deep IR where the transport resides!

This kind of $z \to \infty$ screening behaviour is of course entirely different from the Fermi-liquid wisdoms at work in normal metals. One may contemplate possible consequences. The first one coming to mind is the screening of impurity potentials: short wavelength components of the impurity should be strongly suppressed in the IR while long wavelength com-

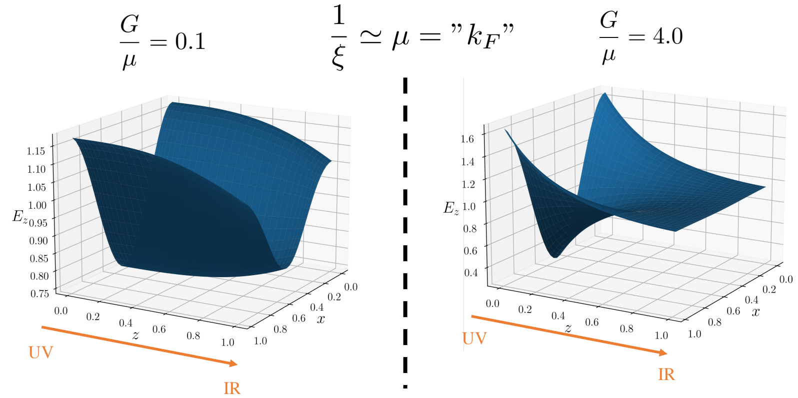

FIG. 13. The electrical field in the bulk as function of radial direction represents the scale dependent effective charge density in the boundary – the geometrization of the RG flow. For $z \to \infty$ one finds that for wave vectors being larger- and smaller than the inverse of the local length the charge modulation in the deep IR is strongly reduced and barely effected, respectively.

ponents survive. This may offer a surprising clue to a long standing puzzle in the cuprates. Potential disorder should be detrimental for d-wave superconducting order because of pair breaking. However, it appears that it is remarkably insensitive to the degree of this disorder. The issue is however that the pair breaking is due to large momentum exchanges tied to the short wavelength components of the impurity potential: when these are suppressed only the forward scattering remains which is harmless for the pairing.

2. The Umklapp momentum relaxation in RN strange metals.

The next issue is, how does such inhomogeneity affect the life time of total momentum and thereby the Drude transport? As in the Fermi-liquid one expects a "memory function logic" insisting that the momentum relaxation rate is coming from an effective coupling between the macroscopic and a momentum sink living at the Umklapp wavevector G,

$$\Gamma = g_G \mathrm{Lim}_{\omega \to 0} \frac{\mathrm{Im} G^R_{TT}(\omega, G)}{\omega} \tag{73}$$

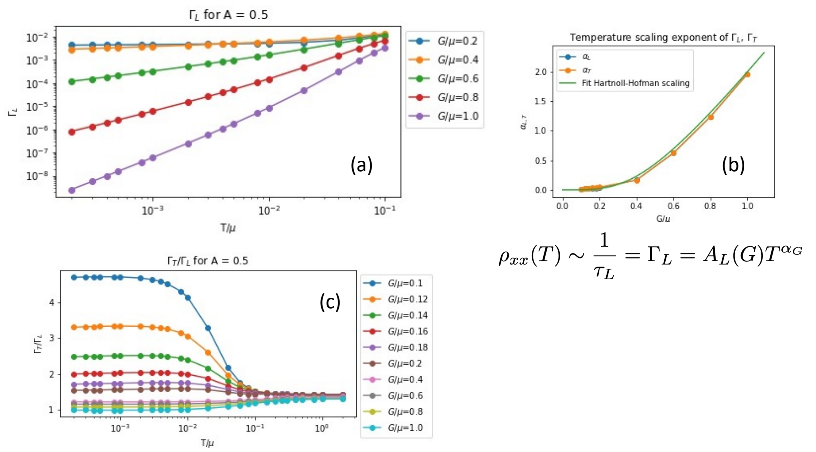

FIG. 14. The momentum relaxation rates associated with the RN strange metal in the square lattice background. (a) The longitudinal momentum relaxation rate Γ_L rate as function of temperature for varying periodicity G. (b) The scaling with temperature follows the predictions for the "generalized Lindhard" deep IR for large G. (c) When the deep IR "crystal lattice" becomes strong for small G's the transversal ("Hall") momentum relaxation rate Γ_T becomes large compared to Γ_L which is associated with the resistivity.

g_G is an effective coupling while the remainder is the slope of spectral function associated with the momentum absorber – this is universally associated with the spatial shear component of energy-stress tensor T_{xy}.

The outcome for the thermoelectric DC transport can be numerically computed. Using all thermo-electric transport coefficients one can use Eq. (68) to separate the incoherent (σ_Q) and Drude contributions: one finds that for $T \leq 0.1\mu$ it is completely dominated by the Drude part and since both the charge- and entropy density Q, s can be computed independently from the thermodynamics one can isolate the momentum relaxation rate Γ – Q and s are approximately temperature independent in this low temperature regime.

Compared to e.g. the way that Umklapp dissipates momentum in a Fermi-liquid the outcome is greatly surprising. The dependence on the strength of the potential A (Eq. 72) is an unremarkable A^2, suggesting that g_G is governed by second order perturbation theory. However, the dependence on G is spectacular, see Fig. (14a) noticing that it is plotted on a log-log scale: the low temperature Γ varies by 6 orders of magnitude! This revolves around a change of behaviour occurring at the wavevector associated with the local length scale $\sim 1/\xi \simeq \mu$.

As suggested early on by Hartnoll and Hofman [68], in the regime that $G \geq 1/\xi$ the momentum sink (G^R in Eq. 73) is associated with the "generalized Lindhard excitations", exhibiting scaling behaviour associated with the scaling geometry realized in the RN near horizon geometry. You already encountered that in the context of the Leiden-MIT fermions in Section (VI E 4). We now need the EOM's of classical fields propagating in this background and the outcome is that for $\omega = 0$ using the ω/T scaling,

$$\Gamma \sim T^{2\nu_G - 1}$$
$$\nu_k = \frac{1}{2}\sqrt{5 + 4k^2 - 4\sqrt{1 + 2k^2}},\ k \geq 1/\xi \tag{74}$$

as for the fermions, one observes that this exponent is in this regime only dependent on the Umklapp momentum and not on θ, z, ζ. This is special for the RN throat. Up to $T \simeq 0.1\mu$ the momentum relaxation of the Drude conductivity closely approaches a perfect power law $\Gamma = A_\Gamma T^{\alpha_G}$. The result is that the computed α_G closely tracks the prediction Eq. (74), see Fig. (14 b). Since $\alpha_G > 0$ in this regime the Umklapp is irrelevant and as in the Fermi-liquid "Galilean invariance is an emergent symmetry at the IR fixed point".

However, when G becomes small compared to $1/\xi$ one finds a completely different behaviour: according to Fig.(14a) the resistivity becomes approximately temperature independent in the Drude regime! This is largely responsible for the big change at low temperature, at $T \simeq 0.1\mu$ the Γ's are varying by no more than an order of magnitude as function of G.

Yet again, we know how this works for sufficiently small G: this is the shear drag regime explained in Section (VII C 2). This is the regime governed by the hydrodynamics and the bulk universality insists that one is dealing with massive gravitons describing the shear drag in the boundary. We can rely on Eq. (71) observing that $\eta \sim s$ (minimal viscosity) and as I repeatedly emphasized RN is suffering from the zero temperature pathology: s is temperature independent and therefore Γ is roughly temperature independent. There is an issue with the length l_η: in general the effective Umklapp in the deep IR may run under renormalization adding temperature dependence to this quantity. But one may then argue that for $z \to \infty$ purely spatial quantities such as l_η should be marginal, temperature independent.

The numerics is just demonstrating that the crossover from "generalized Lindhard" momentum dissipation to the shear drag occurs rather suddenly at a critical Umklapp periodicity set by the local length ξ: at length scales smaller than ξ one is dealing with Eq. (74) switching rather abruptly to the shear drag regime for larger lengths.

The take home message that the influence of Umklapp potentials on momentum relaxation in holographic strange metals is entirely different from the usual Fermi-liquid "particle physics" affair. At the same time we also realize that specifically the RN metal is highly pathological. There is a big need to understand how this works for the arbitrary EMD scaling geometries that I highlighted in Section (VI). Obviously, one has to rely on numerical GR but this at present completely uncharted: the exploration of this vast landscape is the central challenge for the "computational AdS/CMT" effort in Leiden.

For completeness, let me shortly summarize the few other transport wisdoms that have been looked at in this "inhomogeneous" context. This is all restricted to exploratory work restricted to RN set ups which is far from completely understood. Presently these are subject of systematic investigation in Leiden.

3. The Hall momentum relaxation rate.

At stake is in first instance DC transport in magnetic fields in the presence of a lattice. The big deal is that the Lorentz force induces a momentum that is transversal to the electrical field – in fact, it sources angular momentum. This communicates with the *anisotropy* of the spatial manifold associated with the background potential. A crucial aspect is that the relationship between spatial translations and rotations is *semi-direct*: finite translations cannot be distinguished from rotations. But this implies that the effective (in the IR) Umklapp potential becomes weak the isotropy of space restores. Henceforth, such *angular* momentum scattering is inherently non-linear and it can only be addressed with the numerical GR.

It is an easy exercise to show that for a Drude conductor in an anisotropic spatial manifold this just implies that there is a separate "transversal" or "Hall" relaxation rate Γ_T that combines with the cyclotron frequency in the transport coefficients, next the usual "longitudinal" relaxation rate Γ_L highlighted in the above. This is acknowledged in Fermi-liquid kinetic theory where one invokes an "angular" collision integrals to capture the effects. However, the breaking of rotations enters through the anisotropy of the Fermi-surface and the natural outcome is that Γ_T is typically quite similar to Γ_L. This became a big issue in high Tc in the early 1990's when it was discovered that $\Gamma_T \sim T^2$, contrasting with the "Planckian" $\Gamma_L \sim T$ in the optimally doped strange metals.

Prelimenary results indicate that when the IR Umklapp becomes large for small G the Γ_T's deduced from the DC magneto-transport computed for the RN metal on the square lattice become much larger than the Γ_L's : see Figure (14c). The transversal rate Γ_T is deduced by fitting the Hall angle to the Drude expression for this quantity. Upon decreasing the ordering wavevector G the imprint of the lattice in the deep IR is rapidly growing (e.g., Fig. 13). The outcome is that for $G = 0.1\mu$ the transversal rate Γ_T becomes nearly an order of magnitude larger than the longitudinal rate. Different from a Fermi-liquid this unparticle matter reacts strongly to the breaking of the isotropy of the spatial manifold.

But there is yet another universal aspect associated with the "shear-drag" hydrodynamical regime. In hydrodynamics vorticity/circulation is dissipated by the same shear viscosity as the shear in the flow itself. The consequence is that both rates are governed by the same η in Eq. (71). This is clearly reflected in the results shown in Fig. (14c): in the "throat regime" ($T << \mu$) the Γ_T/Γ_L ratio becomes temperature independent.

The behaviour of the Hall angle in near optimally doped cuprate strange metals caused quite some stir in the early 1990's by the observation that $\Gamma_T \sim T^2$, contrasting with the "Planckian" $\Gamma_L \sim T$. Insisting that the transport is of a hydrodynamical nature, given the observation that both angular- and linear motion is governed by the same viscosity, the only way that a different temperature dependence can arise is by invoking a running in the RG sense of the effective Umklapp potential in the deep IR. The relaxation rates are determined in addition by the length l_η. Since the transversal momentum relaxation is strongly non-linearly realized it is natural that it will show a different RG flow than the longitudinal one when the effective potential is itself temperature dependent.

It remains to be seen whether such a behaviour can be realized in a $z \to \infty$ holographic setting. In fact, magnetotransport measurements are playing a key role in the most recent developments: see Section (IX A). On the one hand these supply the best available evidence for gross behaviours that are supporting the quantum supreme matter idea: the strange metal as a *phase* of matter. But at the same time, zooming in on the details of the magneto transport surprising behaviours are found which appear to be also beyond the explanatory power of holographic transport theory in its present state – stay tuned.

4. The optical conductivity in the presence of a lattice.

I already emphasized that the optical conductivity of any system formed from large rest mass UV degrees of freedom such as electrons will show the perfect fluid response where all the spectral weight is in the "delta function at zero". In order to find anything else translational symmetry has to be broken. In the above we focussed on the low energy Drude response but Umklapp has also another consequence in conventional metals: bands are formed and one will find always the "interband transitions" at higher energy. These of course are also responsible for the optical response of conventional semiconductors and insulators: it is associated with "bound charges" that can still form localized dipoles that absorb the radiation.

As I will discuss in the experimental Section (IX), in the cuprates the response associated with the conduction electrons is exhibiting a quite perfect Drude peak at low frequencies. However, getting above 50 meV or so, this is taken over by such a "bound response" which is actually the best branchcut characterized by an anomalous scaling dimension that anybody has ever observed in these systems! From the argument in the previous paragraph it follows as matter of principle that this is somehow originating in the Umklapp: metaphorically, this is like interband transitions by the magic of quantum supremacy have turned into a scaling affair.

Does holography shed any light on this affair? Upon ascending in energy the effective strength of the periodic potential is increasing (the irrelevancy towards the IR) and the non-linearities in the bulk GR will grow: one has to rely on the numerical GR. So much is clear from the little work that has been done that interesting things are going on related to this "unparticle interband transitions" question but this is still rather poorly understood.

Actually, the seminal work showing that the numerical GR could be made to work addressed the (longitudinal) optical conductivity in the RN metal with only an unidirectional periodic potential [69]. These authors picked large G's where the potential is effectively strongly suppressed, but they observed that besides the Drude peak there is a non-Drude "tail" at energies that are still compared to μ: this has nothing to do with the pair creation continuum (σ_Q). They claimed that this was a conformal-like tail but this turned out to be not quite the case.

The only other study [70] also departed from the same set up but looked at what happens for $G < 1/\xi$. Here the effective Umklapp is much stronger and accordingly there is much more weight in this Umklapp induced incoherent part. Sharp resonances develop at $q = 0$, that were interpreted as Umklapp copies of the zero sound mode by these authors. This turns out to be incorrect: we have been studying the optical conductivity as function of *momentum* and this shows that these resonances have nothing to do with the sound mode. At the moment of writing it is completely in the dark what these are.

The take home message of this section is that the understanding of holographic transport in this non-linear regime of strong IR periodic potentials is in a rudimentary stage. At the same time, this is in experiment the most prominent source of information providing evidence for quantum supreme matter physics, see Section (IX). So much is clear that "homogeneous" holography is falling short in providing explanations for the experimental observations, while so little is understood presently regarding inhomogeneous holography that it is impossible to arrive at any definitive conclusion.

VIII. INTERTWINED ORDER AND BLACK HOLES WITH RASTA HAIR.

I already discussed the remarkable mechanism of holographic spontaneous symmetry breaking. I focussed on the first example that was identified: the holographic superconductor which is dual to a black hole with scalar hair (Section VI E 3). But the superconducting order is only the tip of the iceberg. Yet again inspired on top-down set ups, Donos and Gauntlett [71] found out a natural mechanism that leads to the *spontaneous breaking of translations* in holography specifically in two space dimensions. As normal matter is prone to form crystals at low temperature, so do the holographic strange metals.

One anticipates that this again involves technical hardship: also when the cause is spontaneous symmetry breaking these "holographic crystals" require numerical GR in the bulk. Accordingly, the study of such states did not get beyond demonstration of principle and there is surely no systematic understanding presently available.

Yet again, the outcomes are remarkably suggestive in the context of the cuprates. In the underdoped cuprates one can identify the "pseudogap regime" taking over from the strange metal when temperature is lowered (Fig. 1) and during the last quarter century much of the experimental effort went into characterizing electronic ordering phenomena other than the superconductivity that are present here. It developed into a holy grail, the "pseudogap as answer to everything." I am myself of the opinion that this is to a degree a distraction. The fact that it got so much on the foreground is the outcome of a process fuelled by the "economy of science". Meaningful employment is the scarce good in this trade and much of the existent laboratory equipment is geared to detect order. The ensuing employment program gave it more gravitas than it deserves.

It is therefore too much honour for it to be highlighted in the final section that summarizes the experimental situation with regard to "quantum supreme" signals in the data. Let me therefore shortly summarize here the experimental situation. It was a process of discovery: the type of orders that are encountered are in part of an exotic kind that were not known before the late 1980's. In addition there are multiple forms of order that are in a particular fashion synchronized. Metaphorically it is like a "symphony of order parameters", designated as "intertwined order."

Historically, this started out with what is now called the "spin stripes". The early theoretical prediction [41] landed me a Spinoza prize. According to semiclassical mean field (Hartree-Fock) insulating "stripe" domains are formed in a doped Mott-insulator showing the antiferromagnetic spin order of the insulator, separated by magnetic domain walls where the carriers get localized. This is a first example of "intertwinement', the spin and charge order are in an unconventional way in synchrony. This was later followed by claims that are in part still controversial. At the pseudogap temperature T^* (Fig. 1) the onset of *parity* breaking has been claimed, as well as the occurrence of a type of order involving spontaneous diamagnetic "loop currents" that break time reversal (see underneath). The latest addition is the *pair density wave* (PDW): superconducting order at finite wave vector, appearing in synchrony with the charge order.

Despite numerous attempts, it is difficult (if not even impossible) to explain it departing from standard "SRE" semiclassical mean field theory. For instance, there is basically a no go theorem for PDW's both in the weak- and strong coupling limits. Remarkably, already the handful of first attempts to address holographic crystallization reveal that with the exception of spin order (prohibited by the vanishing rest mass) not only the portfolio of exotic orders but also the intertwinement arise naturally as instability of the RN strange metal! It is to a degree even difficult to avoid it. Is this shear coincidence or is so that the quite baroque (”Rasta”) black hole hair associated with this order in the bulk is signalling a deep message? Whatever, let me present here a short overview of this remarkably rich physics of order.

A. Intertwinement by the bulk θ term.

The bottom-up holographic portfolio rests on the principles of effective field theory. As for the Landau free-energy one relies on locality (gradient expansion) and one can add any term based on whatever fields one deems to be of potential relevance that leave the action invariant under symmetry operations. Next to the usual suspects (Hilbert-Einstein, Maxwell, scalar fields, etc.) one also encounters *topological* terms that qualify. In uneven space-time dimensions these are the Chern-Simon terms, famous for being the effective field theory behind the fractional quantum Hall effects in 2+1D. But in even dimensions including our 3+1D universe these take the form of ”theta terms”, famous for the role they play in QCD. In this dimension these are of the form $\sim \theta \varepsilon_{\mu\nu\lambda\sigma} F^{\mu\nu} F^{\lambda\sigma}$. In non-relativistic notation these are of the $\vec{E} \cdot \vec{B}$ form that may be familiar from the unusual electrodynamics in topological band insulators.

The ”axions” have a long history in QCD. This amounts to the idea that the topological angle is determined by a *dynamical* field χ, $\theta \sim \theta(\chi)$. When this complex scalar axion field $\chi = |\chi| \exp i\phi_\chi$ condenses such that $|\chi| \neq 0$ the topological term switches on. By inspecting particular top-down set ups, Gauntlett *et al.* [71] found that such terms naturally arise in the 4D holographic bulk, including specific forms of the potentials. As it turns out, these have as the first effect that the BF ”black hole hair” instability shifts to *finite momentum* corresponding with *crystallization* in the boundary!

But the profundity is still to come. The theta term in the bulk couples three fields: next to the axion χ, it involves also the bulk electrical- and magnetic field strengths. The electrical field is dual to the charge density in the boundary but the magnetic fields encodes for the occurrence of spontaneous *orbital currents*. Here is the origin of the intertwinement that I will illustrate in a moment with a striking visual impression.

But there is yet more room in the bulk: in the axion part of the action one has the freedom to *gauge* the axion field [72] by including a Stückelberg term in the action $(\partial_\mu \phi_\chi - A_\mu)^2$ – in CM this is known as the Josephson action, it turns out that the Swiss theorist Stückelberg

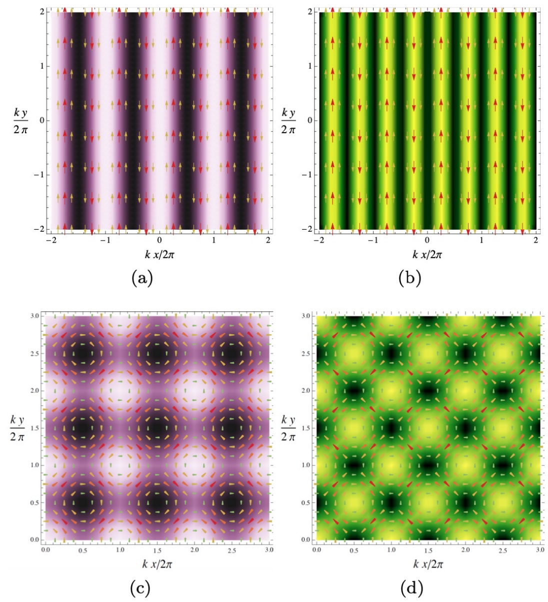

FIG. 15. The holographic intertwined order for a unidirectional- (a,b) and square lattice (c,d) spontaneous crystallization [72] . In panels (b,d) the charge density is indicated by the fals colors In addition the spontaneous orbital currents are indicated by the arrows: the current patterns have a double periodicity as compared to the charge density: for the square lattice this is the same symmetry pattern as for the "d-density waves" in the condensed matter tradition. In (a,c) the amplitude (false colours) and phase (arrows) of the superconducting order parameter is indicated that also doubles the unit cell, being precisely out of phase with the diamagnetic currents.

figured this out already in 1938. This has the effect that at the moment the axion condenses holographic superconductivity sets in, albeit at finite momentum as well. This is the pair density wave!

It is not only the case that all these orders set in simultaneously at T_c – the parity breaking being the primary order parameter – but these are also intertwined in the cuprate sense of the word. This is illustrated in Fig. (15). One can as well address a undirectional- ("stripy") or a square lattice translational symmetry breaking. The green false colours in the right panels indicate the charge density modulation, the "crystal" in the literal sense. As most easily inferred from the unidirectional case, the arrows in the left panels indicate the spontaneous currents that have twice the periodicity of the charge modulation. The purple false colours indicate the *pair density wave* having the same periodicity as the currents, being however precisely out of phase. This is just like the "spin stripes", with the currents and the PDW taking the role of the antiferromagnet. The same logic governs the square lattice where now the currents form closed loops, ordering in a similar way as envisioned in the loop current claims.

Surely, this has an image in the bulk in the form of a highly textured "corrugated" black hole hair in the bulk: this "black hole with Rasta hair" is the most complex fanciful stationary black hole solution known to mankind! Is it shear coincidence that it gets so close to this similarly unprecedented "order circus" encountered in the cuprates? This is so intriguing that I allow myself to be a bit religious just for the pleasure of it – it is the heavens that speak to us in a piece of rusted copper ...

B. The holographic Mott insulators.

It is not yet the end of the story: when it comes to black hole complexity one can shift to even a higher gear. I discussed the crucial role that the Mott insulator plays in the high Tc problem in Section (IV D). The industry standard is to depart from Hubbard type models like Eq. (42). But these are quite specific, in fact quite oversimplified models that were introduced as toy models being taken in the mean time more literal than intended when these were formulated. The Mott insulator is actually resting on a much more general principle that is independent of microscopic modelling. In full generality, the electron Mott insulator is a *crystal formed from electrons with a crystal structure that is commensurate with the underlying ionic lattice*.

It just examplifies the general idea of *commensurate pinning* that is also occurring in meat-and-potatoes classical systems. It is very simple. Form a "balls and springs" solid with a particular lattice structure and lattice constants. In the spatial continuum this will be characterized by its Goldstone bosons – the massless phonons. But embed it now in a background containing a potential that precisely fits the spontaneous crystal lattice, and this will shift in such away that its "atoms" are lying precisely in the external potential minima. The effect will be that the phonons will acquire a *commensuration gap*, associated with the energy required to rip the crystal from the background. High energy physicist know this general phenomenon as the "pseudo-Goldstone bosons", referring to the general behaviour of Goldstone bosons in an explicit symmetry breaking field.

The ensuing state is clearly insulating, characterized by the commensuration gap which is a better name than the "Mott gap". Moreover, symmetry wise it is a featureless state because the external potential has already explicitly broken the translational symmetry. The only additional ingredient one meets dealing with electrons are the spins at low energy.

This suggests a window of opportunity to construct Mott-insulators in holography. We just learned that holographic matter can be tailored to crystallize. All one needs is to switch the external lattice potentials of the previous section and see what happens. This was accomplished in Leiden, with Sasha Krikun in the drivers seat [73]. For technical reasons the focus was on a unidirectional lattice – in the mean time we also got it to work for a square lattice pushing the bulk numerical GR. The outcome is that it works precisely as anticipated. In essence it is like the intertwined order (without PDW's) where the commensurate background potential acts to further enhance the charge modulation.

A difference with the Hubbard model variety is that because of its ultrarelativistic nature holographic matter does not support spins that can go their own way. The role of the spins is taken instead by the orbital loops and we identified the analogue of superexchange finding that it takes a much smaller energy to "flip" the orbital loops than the charge gap.

But there is more complexity around the corner. The "spin stripes" I advertised in the above should be viewed in full generality as "higher order commensurate Mott-insulators". This is referring to well established principle in a subject that was big in the 1970's: the study of crystals having a periodicity that does not quite match the background potential. Imagine a mismatch of e.g. a quarter of a lattice constant. One can formulate ball-and-spring models in such a background ("Frenkel-Kontorova models"). These are intrinsically non-linear and one finds typically the formation of soliton like textures. These are called *discommensurations*: the crystal stays locally commensurate, forming commensurate domains, while the mismatch is stored in narrow areas (the discommensurations) that form a regular discommensuration lattice. This is precisely describing the Hartree-Fock spin stripes, called like this in the first Zaanen-Gunnarsson paper, referring in fact to classical "striped" discommensuration patterns.

Especially dealing with a one dimensional modulation, the "Devil's staircase" was identified in such systems. Vary the commensurate mismatch continuously. At various rational

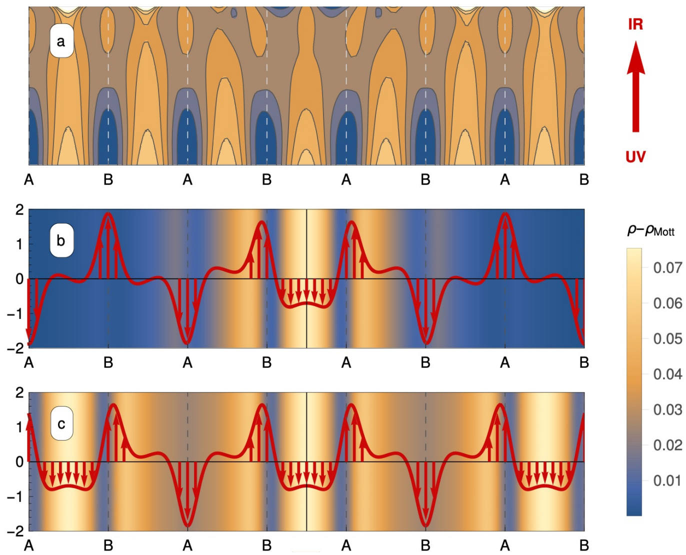

FIG. 16. The "stripes" in the holographic Mott insulator [73]. When the external- and spontaneous translational symmetry breaking are incommensurate discommensurations form. In (a,b) we show the structure of one such discommensuration associated with a unidirectional symmetry breaking. (a) The electrical field in the bulk as function of radial direction yielding the RG flow associated with pinning and the formation of the discommensuration in the middle. (b) The difference of the charge density compared to the commensurate Mott insulator (false colors) and the current order (red line, arrows) revealing that the discommensuration is a domain wall in the staggered current order. (c) These "stripes" form a regular pattern, shown here for a three lattice constant discommensuration lattice.

ratio's (like the 1/4 of the example) a higher order commensuration point is realized and a discommensuration lattice forms. But now change the ratio a bit: what happens is that a discommensuration lattice forms inside the commensuration lattice. And so forth, the outcome is a fractal hierarchy of such discommensuration lattices, the Devil's staircase. Amazingly, such a Devil's staircase can be realized departing from the holographic matter!

For the Hubbard-Hartree-Fock stripes the charge discommensurations are at the same time domain walls in the antiferromagnet formed by the spin system. Amazingly, the same is happening in the "holographic stripes" but now involving the orbital currents. These form an antiferromagnetic pattern in case of the spontaneous crystal where the direction of the current flips traversing a maximum in the charge density, see Fig. (15). In Fig. (16) the outcomes are shown for the "stripes" in the holographic Mott insulator. We depart from a unidirectional translational symmetry breaking as in Fig. (15b) adding a similar unidirectional potential which is however incommensurate. In panels (a,b) we focus in on a single discommensuration. In panel (a) the electrical field in the bulk is shown as in Fig. (13) providing an image of the RG flow but now related to the pinning physics. The primary modulation wavevector is $\sim \mu$ and the external potential would by itself die-off in the IR. But the spontaneous symmetry breaking is relevant to the IR and one see from the figure that these get clamped together with each other midway the radial direction: the RG view on commensurate pinning!

But this clamping is interrupted midway the space direction: the discommensuration. Focussing on the charge density (false colors in panels b,c) the charge does indeed accummulate at the discommensuration, Focussing now on the currents (arrows) one discerns that the discommensuration corresponds at the same time with a *domain wall* in the staggered current order. This is precisely what is happening in the cuprate spin stripes, after identifying the staggered antiferromagnetism with the staggered current order. Surely, these stripes form together an orderly striped structure, Fig. (16). The bulk incarnation in the form of the hairy black hole sets the complexity record for stationary black holes!

What happens with the excitation spectrum in the holographic Mott insulator? The optical conductivity shows that the Drude peak associated with the sliding mode in the continuum shifts up to a finite energy: this is just the expected pinning "Mott" gap. I already discussed at length the generic "incoherent" second sector in Section (VII B 3), with a big warning sign given that it is a specialty of ultrarelativistic matter. However, we are now dealing with a deep IR/near horizon geometry that is grossly reconstructed because of the highly non-linear impact in the bulk of the translational symmetry breaking. The outcome is that although the Drude conduction is completely gapped an incoherent sector is left behind at low energy showing the typical power law conductivity of such incoherent stuff, with a resistivity that is power law divergent towards zero temperature.

As I emphasized in Section (VII B), *emergent charge conjugation symmetry* is a necessary condition for having finite conductivity exempted from a Drude transport. The most direct, simple observable is that the *Hall effect will vanish*. This is very easy to see. Consider two equally large basins with opposite charge and apply an electrical field and a magnetic field in a perpendicular direction. Due to the Lorentz force as may positive- as negative charges will rain down on the boundary and no Hall voltage will arise. To check this a real two dimensional lattice is required. In the mean time we got this working and we find that indeed the Hall effect vanishes upon entering the holographic Mott insulator.

This is directly communicating with highly serendipitous recent experimental results by Popovic and coworkers on the cuprate spin stripes [74]. These show typically a remnant superconductivity that can however be completely removed in magnetic fields ~ 30 Tesla which are available at the high field labs. They find that in this "cleaned up" spin stripe phase a slowly diverging resistivity in tune with the expectations for the incoherent stuff. But more strikingly, *the Hall effect is abruptly disappearing* upon entering this phase. This may be the first and best example of emergent quantum supreme matter that persists down to the lowest temperatures. This begs for a large experimental effort, measuring as many physical properties as possible in these difficult circumstances of very large magnetic fields and ultra low temperatures.

IX. THE FOG OF WAR: THE EXPERIMENTAL SITUATION.

The specific appeal of this whole affair is to find out whether quantum supreme matter is realized in nature. The high Tc superconducting cuprates take in this regard the role of fruit fly. Above all these are the "ground zero" of the intellectual crisis that developed during the last thirty years, where it became increasingly clear that the established paradigm of "classical matter" (the ESR products) fails in explaining a large body of experimental findings. In a parallel development unfolding in the computational community we became increasingly aware of the fundamental brick wall in the form of the fermion sign problem as highlighted in Section (IV). We better accept the fact that the mathematical tool box required to decode the way in which nature works in this realm used to be completely empty.

This has now changed by the arrival of AdS/CMT. But to my opinion it is not quite reliable as a navigational device. I like to put forward the following metaphor. Imagine it would have been extremely hard to measure the electrical- and magnetic properties of aluminum metal at sub-Kelvin temperatures, while it would have been easy to measure up the stuff in the core of a big neutron star. It is not certain what is happening out there but with a little luck these experimentalists would have discovered that the quarks form at these high densities a rather weakly coupled Fermi-liquid that upon further cooling would turn into a flavor-colour locked superconductor: another invention by Wilczek, it is a BCS superconductor driven by the attractive nature of gluon exchange, accommodating the zoo of quantum numbers of the standard model in an unconventional (in the BCS sense) order parameter.

But there is then obviously a long way to go all the way to aluminium, finding out that electrons have not much of internal symmetry as compared to the zoo of falovors and colors of QCD. Instead, there is bandstructure, that the electrons exchange phonons that for strange reasons (from the QCD perspective) give rise to attractions that can even overwhelm the huge Coulomb repulsions. But both cases are governed by the same overarching emergence 'meta' principles. There are Fermi-surfaces and quasiparticles in the normal state, the condensates are governed by the principles of spontaneous symmetry breaking, etcetrea. In this analogy, AdS/CFT is like the neutron star matter while high Tc-related phenomena is like aluminium.

The limitations of the AdS/CFT navigator are obvious. Instead of dealing with the zero density CFT "parts" that at finite density turn into the holographic strange metal wholeness, one departs from the condensed matter electrons that by being subjected to the strong lattice potential (the "Mottness") may be forced into the quantum supremacy realms. I already highlighted various manifestations of "UV sensitivity" that cause matters to become different by principle. But it may well be deeper. The central principle revealed by AdS/CMT is the covariant scaling. I am a believer in the universality of θ and z anomalous dimensions (see underneath) but "covariant RG" is much less constraining than the full fledge invariance under scale transformation which is at the heart of the notion of the stoquastic critical state universality.

The first question to ask is, is it really sure that the cuprate electrons are not submitting to the established canon of "classical" matter? For most of its history it was taken for granted that some contrived version of the meat-and-potato diagrams would eventually explain everything. Although large parts of the high Tc community is still living in this tunnel, the data that have been streaming in very recently make this proposition in the mean time untenable, as I will discuss underneath. These new results are suggestive othe "meta" emergence principles suggested by AdS/CMT to be at work in the cuprates: (1) The strange metals appear to be *phases of matter* in the same general sense as Fermi-liquids are (Section IX A), (2) These appear to be governed by the covariant scaling principle revealing extremely anomalous dimensions (Section IX B) while the finite T transport reveals the Planckian dissipation, (3) Although photoemission has been historically the experimental source for the (non-quantum supreme) presence of particles, very new results indicate that this is actually delusional (Section IX C).

But ironically, these gross properties are revealed through experimental signals that to quite a degree are *not at all explained by AdS/CMT* in its present state, as I will emphasize in the discussion. It may be that these reside in the "inhomogeneous", strongly non-linear bulk regime as discussed in Sections (VII, VIII) that are barely explored. But I would not be surprised when eventually it will turn out that the cuprates are in a quite different "covariant universality class" where phenomena occur that are unheard off in the "QCD-like" version. To be continued ...

A. Cuprate strange metals: quantum critical phase versus quantum critical point.

Already since the 1990's the idea that the "strangeness" of strange metals originates in the presence of a quantum critical point somehow related to the Hertz-Millis ploy explained in Section (III D) has been quite influential. There are quite a number of examples especially in the heavy fermion family where the basic idea is very credible [24]. One does find (mostly) magnetic order parameters that do undergo a zero temperature transition. A "quantum critical wedge" showing anomalous properties is anchored at the quantum critical point. The associated cross-over lines in the coupling constant-temperature plane are clearly discernible, even showing the tell tale signs of (heavy) Fermi-liquid re-emerging in the renormalized classical regime. As I already emphasized, this does not imply that these are fully understood but there is no doubt that the "strangeness" is eventually rooted in the QCP.

Until rather recently it was a community consensus that the cuprates were also in the grip of a quantum phase transition of the kind at optimal doping. The phase diagram published as part of the " community consensus review" that appeared in Nature in 2015 [1] is reflecting this, see Fig. (1). The purple "strange metal" area is indicated as the wedge. On the underdoped side the T^* line decreases roughly linearly with doping, suggestive that it hits zero temperature at the "critical doping" p_c. The reflex has been to associate pseudogap temperature T^* that reveals itself in a form of incomplete gapping affecting nearly all physical properties with the onset of order at a classical phase transition. But this continues to be a confusing issue – it is far from settled whether this directly involves the intertwined order package that I shortly discussed in Section (VIII).

A difficulty is that as far as one can say all these orders have disappeared at dopings that are significantly lower than p_c. It is still impossible to complete surpress the superconductivity near optimal doping because of the extremely high magnetic fields that are required to get above the (mean field) H_{c2}. One therefore has to look for the signatures of the vanishing order deep in the superconducting state but despite intense effort nothing was ever detected. One idea is that the order parameter is just very hard to detect with standard experimental machinery: the "hidden order" – the loop currents presented in Section (VIII A) are an example.

All along there was however an ideological bias. Until the arrival of the holographic strange metals all that could be imagined was the QCP. At least in my mind there was all along an uneasiness. In so far such metallic QCP's are tractable (in essence, implicitly departing from a Fermi-gas plus order parameter) at the end of the day the quantum critical behaviour is an infrared affair. Upon rising energy or temperature a regime which is Fermi-liquid like should re-emerge: this is typically observed in the heavy fermion QCP systems. But not so in the cuprates: for instance, the linear "Planckian" electrical resistivity extends all the way to the temperatures where the crystals melt.

Another uneasy affair was the lacking of any experimental signature of the crossover line in the overdoped regime for $p > p_c$. The crossover from the strange metal (purple) to a presumably Fermi-liquid like overdoped regime (white) is just a product of imagination: it is not supported by literally *any* empirical support. The experimental characterization of the metals in the overdoped regime has been spectacularly on the move since 2015. In a remarkable pace, evidences accumulated supporting a quite different basic view than the traditional "QCP".

Apparently, at dopings $p < p_c$ a first type of strange metal phase is realized that is prone to become "BCS-like" unstable to the intertwined order. It is BCS-like in the following sense. Intrinsic to quantum phase transitions is the general notion of "fluctuating order". Upon zooming in on the dynamical susceptibility associated with the order parameter one should find strong enhancements and the E/T scaling and so forth as discussed in Section (III) in the metallic state up to the temperatures where the metal is "strange". However, in a weakly coupled BCS superconductor (or equivalently, a "nesting" type density wave instability) such correlations disappear at temperatures well above the thermal transition. Only upon closely approaching this transition the RPA ("paramagnon") style relaxational peak develops. With the caveat that the experimental probes are far from perfect, nobody has managed to pick up signals of the quantum critical fluctuating order kind over the large temperature range where the "linear in T resistivity" strange metal is realized. This is supportive of a "weak coupling like" development of the order – its correlations are rapidly petering out above the transition temperature in the strange metal. Regardless whether it is merely metaphoric or more literal, the holographic answers are enlightening. Realizing the generalized Fermi-liquid and the quantum critical BCS of Section (VI E) one learns that this weak coupling-like behaviour supersedes the Fermi gas. In addition, the take home message of the "Rasta hair" of Section (VIII) is that very different from the Fermi gas such weak coupling like instabilities may support complex, multi order parameter instabilities.

The news that broke in 2019 is the experimental discovery [75] of a *discontinuous* first order like zero temperature phase transition that appears to take place at p_c. The discoveries that followed leave no doubt that a strange metal phase of a different kind takes over in the overdoped regime. Eventually, beyond the end point of the superconducting dome there are signs that this merges into a Fermi-liquid like affair, see Fig. (17).

This transition is by itself most unusual: even holography has presently nothing to say about it. It was first seen in ARPES [75] in the form of a dramatic change occurring at rather high energies in the electron spectral functions. Below p_c in the normal state the spectral functions associated with the "antinodal" momentum directions are completely incoherent "unparticle" affairs. Above p_c this suddenly changes, showing quasiparticle signatures – whether these are real quasiparticles is a different issue, see underneath. This was quickly followed by specific heat evidence for the sudden collapse of the remnant pseudogap upon crossing into the overdoped state (also supported by ARPES). Instead of the "pseudogap" order to be the culprit, the moral is that the underdoped metal is prone to the BCS-like "pseudogap" instability while the overdoped metal is only unstable in the superconducting pair channel.

The highly unusual aspect of this "first order like" change is that many macroscopic

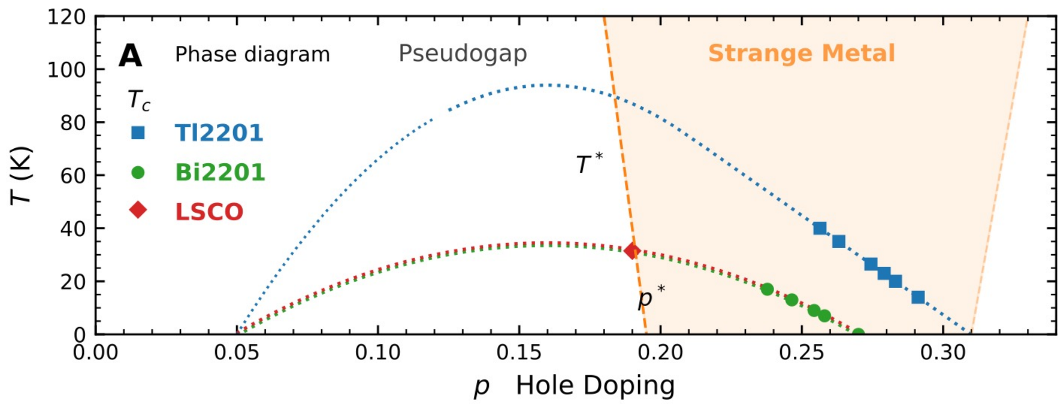

FIG. 17. The high Tc phasediagram updated on basis of magnetotransport measurements [76] emphasizing that the metal in the overdoped regime is a *phase* of matter, different from the metallic phase below optimal doping (p_c) that is prone to "pseudogap instability" at low temperatures.

properties such as the superconducting T_c, the resistivity and so forth are varying in a perfectly smooth manner around p_c: these are expected to jump upon crossing a zero temperature first order transition. However, very recently it became clear that such signatures are revealed in *magneto* electrical transport properties. A discontinuous change has been observed in the Hall relaxation time at p_c [77], supported by magneto-resistance measurements. The latter [76] supply direct evidence for the strangeness of the overdoped metal, in the form of the "Planckian quadrature": the magneto-resistance appears to scale with $\Delta\rho \sim k_B T \sqrt{1 + (\mu_B B / k_B T)^2}$ (B is the magnetic field strength), a poorly understood "Planckian" scaling behaviour also observed in a QCP system. Yet, in the overdoped metal it occurs in the whole doping range: it is a property of this metallic *phase*. Unpublished results by Hussey's group show that this Planckian quadrature magnetoresistance suddenly disappears below p_c being supplemented by a Drude type "Kohler's rule", i.e. governed by the $\Gamma_{L,T}$ discussed in Section (VII D 3).

I perceive this as the best empirical evidence for the cuprate metals to be forms of quantum supreme matter. Ironically, it resolves the ideological divide that has haunted the subject since the very beginning. On the one hand, on the gross level the physics appears to be similar to the Fermi-liquid affair and a large part of the community found in this comfort to run variations on fermiology to explain the observations. However, a better educated (mostly theoretical) company struggled with all kind of inconsistencies. Upon getting informed by holography regarding the existence of the generalized Fermi-liquids hidden "behind the fermion sign brick wall" this ambiguity evaporates.

But this does not mean that one can read off all the answers from the holographic handbook. Holography has at this moment in time *nothing* to say about the "strange" first order transition at p_c. Neither does it shed any light on the difference between the underdoped- and overdoped strange metals. Although holography is revealing with regard to the "Hall angle" question (Section VII D 3) it does not tell anything about the "quadrature" magnetoresistance. Yet again there is irony: holography does not shed *any* light on the magneto-transport and ARPES data that reveal the separate existence of two types of strange metals.

B. The collective dynamical responses: currents and charge density.

More is needed to find out whether these strange metals have anything to do with the quantum supreme matter of holography. As I emphasized repeatedly, holographic metals can be viewed as "generalized Fermi-liquid", with a gross structure that it shares with the Fermi liquid but characterized by anomalous scaling dimensions. To find out about the scaling structure associated with such quantum stuff one needs dynamical information. From the discussion in the previous sections you learned that one needs the dynamical response functions in principle over a large energy, temperature and momentum range to find out whether and how these properties are scaling, to subsequently determine whether the specific dimensions suggested by holography are at work. For mostly practical reasons such data are still rather scarce. Let me present here a short overview.

Transport has gotten much emphasis for the simple reason that it is readily available in the laboratory. But for purposes of discerning scaling laws it is less ideal. As I explained in Section (VII) it is typically governed by conservation laws, in the first place the conservation of total momentum. The "strangeness" enters through the "weak" violation of the conserved momentum in the first place by the presence of the periodic lattice. One in fact learns how this is "filtered" by the cooperative properties of the strange electron fluid as I discussed at length in Section (VII).

In fact, to obtain an "unfiltered" view on scaling properties one would like to measure properties that are *not* protected by conservation laws. A first example that you encountered is the dynamical pair susceptibility that is central to the superconductivity. As I explained in Section.(VI E 3) this exhibits scaling in the Fermi-gas (the "Cooper logarithm"), predicted to pick up anomalous dimensions in the strange metal. But this is practically impossible to measure and the same is the case for nearly all other non-conserved operators that come to mind. For instance, the spin fluctuations at large momenta which can be measured by inelastic neutron scattering qualify but because of the bad signal to noise ratio these cannot be resolved in the strange metal regime. In fact, the very recent EELS experiments measuring the charge dynamical susceptibility that I will highlight underneath are at the moment stand alone in this regard.

1. The linear resistivity and the Drude behaviour.

I have already repeatedly referred to the oldest and in a way still most obvious indication that something highly unusual is going on: the DC resistivity being linear in temperature all the way from T_c to temperatures where the crystals melt (e.g., Section VII C 2). The difficulty is to explain why it is so *simple*. In a Fermi-liquid the temperature dependence of the resistivity is governed by the way in which individual quasiparticles loose their momentum. By principle this is a strongly temperature dependent affair: the "Umklapp T^2" at the lowest temperatures, taken over by the phonons that can cause a variety of temperature dependencies to eventually saturate when the inelastic mean free path becomes of order of the lattice constant. Especially the violation of the latter Mott-Ioffe-Regel (MIR) limit is striking. Around 400K or so this is reached in the cuprates but there is no sign of it in the data – the resistivity stays perfectly linear.

To find out what is going on *dynamical* information – optical conductivity – is crucial also in this case. From DC transport it is just impossible to pin point the general nature of the transport physics and one encounters frequently misinterpretations. The optical conductivity

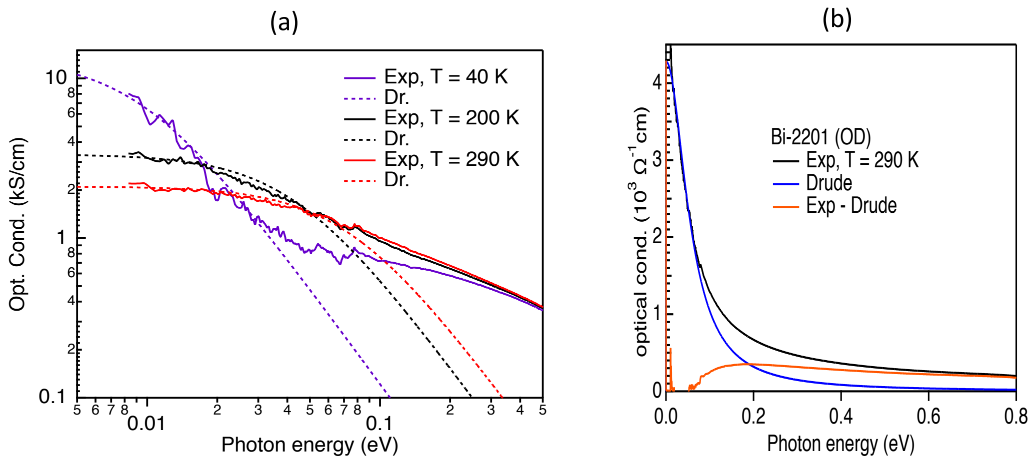

FIG. 18. The real part of the optical conductivity of cuprate metals [78]. (a) The low frequency part is perfectly fitted with a simple Drude form over a large range of temperatures and doping. (b) After subtracting this Drude part (blue) of the overall signal (black) a contribution is left over that is a perfect branch cut (red) above a gap ~ 50 meV.

measurements leave no doubt: at low frequencies these reveal a textbook Drude lineshape, Eq. (63). This was already noticed when the first measurements became available in the late 1980's. Pushing the limits of the effective resolution in a recent study [78], it was concluded that at least 90% of the low energy spectral weight is at least at temperatures below the MIR captured by the Drude conduction, see Fig.(18).

The DC conductivity is then the product of the Drude weight and the momentum relaxation rate $\sigma(T, \omega = 0) = \mathcal{D}\tau_P$. The only way to separate these two factors is again by direct measurement of the frequency dependence – studies that only use DC information guessing the Drude weight based on Fermi-gas relations are just not to be trusted. It follows from this optical study that $\mathcal{D}$ is nearly temperature dependent while it increases in a linear fashion over the doping range all the way from mildly underdoped up to the end of the overdoped regime, varying smoothly at p_c: claims that there would be sudden change in carrier density based on Hall effect measurements are flawed – these changes are actually happening in the Hall relaxation rate Γ_T [77].

The linear resistivity is therefore entirely due to the momentum relaxation rate, and this can be shown to be very close to the Planckian dissipation time $\tau_p \simeq \hbar/(k_B T)$ [78]. I highlighted this as the natural time associated with the "conversion of work in heat" in densely entangled systems. The difficulty is however that it is now governing the life time of total momentum and by hydrodynamical principle this is only finite when the "deep IR" translational symmetry is broken. By default it cannot be truly universal, it is somehow tied to the way that the translational symmetry breaking is "processed" by the strange metal: this is just implied by the fact that it is a Drude transport.

The only available internally consistent explanation I am aware of is the minimal viscosity "shear drag" suggested by holography, Section (VII C 2). It is far from proven fact – the first feature that one would like to observe in the laboratory is that the electron system behaves like a hydrodynamical fluid but this is a tall marching order for the experimentalists. Another difficulty is related to the phonons: upon raising temperature these become available as extra momentum sink next to the Umklapp scattering, and why do these not imprint on the temperature dependence? Even more pressing, above the MIR temperature it is quite questionable whether one is dealing with a Drude conductor [79] and yet again the issue is why nothing is seen in the data crossing over from simple shear drag at low temperature to something else in the "bad metal" high temperature regime.

2. Optical conductivity and the branch cut.

But the optical conductivity reveals more: in fact the most perfect branch cut response signalling scaling takes over from the Drude response in the energy range ~ 50 meV all the way to the very high energy of roughly 1 eV [80] . This is illustrated in Fig. (18) [78]: the low energy part can be highly accurately fitted with a Drude peak with a frequency independent Γ. Upon subtracting this from the measured σ_1 a higher energy part is left over that can be fitted with a branch cut characterized by a smeared low energy gap,

$$\sigma_{\exp}(\omega) = \sigma_D(\omega) + \sigma_{\text{incoh}}$$
$$\sigma_D(\omega) = \frac{\mathcal{D}_D}{\Gamma - i\omega}$$
$$\sigma_{\text{incoh}} = \mathcal{D}_{\text{incoh}} \left(\sqrt{\Delta^2_{\text{incoh}} - \omega^2} - i\Gamma_{\text{incoh}} \right)^{-\alpha_{\text{incoh}}} \tag{75}$$

Here $\Delta_{\text{incoh}} \simeq 0.05$ meV and Γ_{incoh} parametrizes the smeared gap at the low energy end. Although the modelling of the cross over from the Drude to the branch cut part is ambiguous it is crucial that it both fits the real- and imaginary part of the measured conductivity. At energies well above Δ the branch cut takes over and this signals itself through a frequency independent phase angle that lights up in the data : $1/(i\omega)^\alpha = \exp\left(-i\alpha\pi/2\right)/|\omega|^\alpha$.

The scaling dimension α_{incoh} shows the signs of qualifying as a genuine anomalous dimension. At optimally doping it is approximately 2/3. However, in this recent study [78] the optical conductivity is systematically investigated in a large doping regime. Further evidence for the overdoped metal above $p_c \simeq 0.19$ continuing to be strange is in the finding that the "conformal tail" extends all the way to the highest doping while its relative spectral weight is barely changing. But these data indicate that α_{incoh} is continuously varying as function of doping up, varying in the single layer BISCO from $\simeq 0.4$ at very low doping $p \simeq 0.05$ to $\simeq 0.8$ at the highest dopings where superconductivity has disappeared.

Such branch cuts with varying exponents are suggestive that we are indeed dealing with a quantum supreme metal in the spirit of the holographic strange metals. But there is still quite some distance to go. As I emphasized in Section (VII) the presence of the ionic lattice is a *necessary* condition to find this optical spectral weight. I also explained in Section (VII D 4) that presently it is still to be found out whether general principle can be extracted for such spectral weight from holography, since "inhomogeneous" holography is still poorly understood.

Especially with regard to this "optical branch cut" I would not be surprised when eventually it turns out that this is beyond the capacity of holography. After all, the Umklapp that is at the origin of the branch cut is in the electron systems the key ingredient behind the "Mottness" that is the prime suspect for the "release" of the fermion sign driven quantum complexity as explained in Section (IV D). A strikingly mysterious fact is that the branch cut appears to extent up to energies as high as 1 eV where one gets close to the UV physics of the electrons in Cu-O unit cells. This invokes UV sensitivity – I would not be surprised when it will eventually become clear that although the general scaling principles revealed by holography are applying we are dealing in the condensed matter systems with a different universality class of the covariant scaling agenda than the ones revealed by the large N CFT's.

3. EELS and the local quantum critical charge response.

This brings us to the pioneering, recent effort to measure the dynamical charge susceptibility using electron energy loss spectroscopy (EELS) [81]. Obviously, the first object one would like to know as a theorist is this very basic property, especially since this spectroscopic technique makes it possible in principle to measure the response over basically the whole parameter range of frequency, temperature and especially also momentum as of relevance to the condensed matter electrons. One can wonder why this was not accomplished already 25 years ago. This is because of politics. EELS is a rather old technique and it was hard to get funding for the necessary upgrades. After the success of the Abbamonte group this seems now to be quite different – with a little luck there will be soon much more data of a higher quality. One should also keep an eye on the Resonant Inelastic X-ray Scattering (RIXS). This technique is characterized by a million fold or so larger carbon footprint but it is at the instrument builders frontier involving gigantic beamlines at last generation synchrotrons. The difficulty is that it reveals much more "dirty" information – it is not a simple linear response affair and it is still littered with interpretational difficulties.

The main outcome of the pioneering EELS experiment is shown in Fig. (19) obtained in an optimally doped cuprate [81]. This should be directly compared with the (longitudinal) prediction for the spectral function associated with dynamical charge susceptibility of the Fermi-liquid with the plasmon and especially the Lindhard continuum. It looks entirely different: instead of the Lindhard continuum with its strongly dispersing upper bound as function of momentum and linear frequency dependence (Section VI E 2, Fig. 11) one finds a continuum that is (a) *independent* of frequency up to a cut off ~ 1 eV (the "marginal" frequency scaling dimension) and especially (b) it appears to be entirely momentum independent up to momenta halfway the Brillouin zone as highlighted by the scaling collapse in the figure.

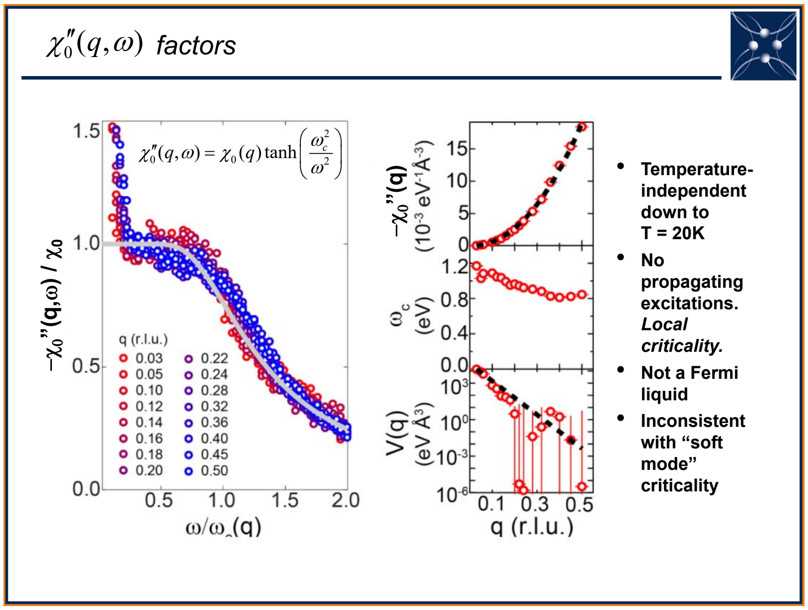

FIG. 19. The electron loss function in an optimally doped cuprate, in a large range of frequencies and momenta [81]. The data show an impeccable scaling collapse for all measured momenta, demonstrating the local quantum criticality governing the charge density response.The frequency dependence is is marginal – frequency independent – up to a UV cut off $\omega_c \simeq 1$ eV.

This is the direct experimental confirmation of the claim that the cuprate strange metal is local quantum critical, the $z \to \infty$ scaling acting prominently in Section (VI). I perceive this as the most direct evidence for the cuprate strange metals to be of a holography-type quantum supreme nature. Such scaling behaviour is surely beyond the realms of semiclassical "particle physics" while it is also beyond what is possible in stoquastic quantum critical states. I remind the reader of the general scaling arguments explained in Section (VI D) insisting that the hyperscaling violation exponent θ has to come to rescue to prevent the zero temperature entropy to take over. The diverging dynamical critical exponent is just the anomalous dimension of choice to once and for all disqualify quasiparticles.

Yet there are still quite some issues that are far from settled. How does it relate to the branchcut found in the same energy range in the optical conductivity? This would translate via the continuity equation into a (transversal) polarization propagator scaling like $1/(i\omega)^{1+\alpha}$ at $q = 0$, quite different from the marginal scaling suggested by the EELS data. The dogma is that at $q = 0$ one moves the electron fluid uniformly and it is impossible to discern a different transversal- and longitudinal response. The only loop hole I knowis associated with the goldstone bosons: when there is structure in the unit cell the acoustic phonons are subjected to Umklapp and optical phonons appear at zero wave-vector. It is of course well known that the transversal- and longitudinal optical phonons are different at zero momentum. I emphasized in Section (VIII B) that the Mott insulator is fundamentally an electron crystal where the acoustic phonons are gapped by commensurate pinning. Could it be that the observation of a very different transversal- and longitudinal response is somehow anchored in the optical phonons of the electron crystal? In other words, is this a hitherto unrecognized ramification of the Mottness of Section (IV D) persisting in the metallic phase?

Finally, the first EELS results on the doping dependence are also surprising [82] , emphasizing further the differences between the transversal and longitudinal response. The claim is that at high temperatures it all looks the same as for optimal doping. But upon lowering temperature changes are seen as function of doping. The momentum independence is within the resolution of the experiment not affected, but in the underdoped regime the authors find that at lower temperatures the charge susceptibility undergoes a reorganization on a large energy scale. The lower energy part changes drastically, showing signs of becoming relevant in the same sense as discussed for the holographic superconductivity (Section VI E 3), bending upwards as function of decreasing energy. But this makes sense, just indicating that an instability towards charge order starts to develop. The intriguing part is that this seems to happen over a large momentum range: apparently that the "stripy" charge order wavevectors are picked up out is a last minute affair. The temperature evolution in the overdoped regime is the real surprise: upon lowering temperature a "hole" appears in the low energy spectral weight. This adds further mystery to the overdoped strange metal: one would expect to see this scale-full behaviour to imprint on the deep IR but nothing is seen in the DC data at the characteristic temperature indicated by the EELS data. Altogether the big deal is that theE/T scaling is grossly violated: these changes happen on an energy scale $\sim$ 0.5 eV which are very large as compared to the temperatures $\sim$ 100 K where it sets in both on the under- and overdoped side. This is reminiscent of the "spectral weight transfer" effects that are tied to Mottness.

I expect further advances in a near future on this frontier. The results of the Abbamonte group are obtained using a low energy reflection mode EELS machinery. This has its disadvantages as compared to the very high energy transmission EELS. The latter is an old technique, that used to have a bad energy resolution of order of 0.5 eV. But with modern electronic optics this can be much improved and presently several groups are building such new rigs inspired by the success of Abbamonte's machine.

C. The fermions: ARPES and STS.

Considering what experimentalists can do in the lab, the only spectroscopic techniques that deliver the required dynamical information with astonishing momentum, energy and temperature resolution are the "fermionic" photoemission (ARPES) and scanning tunneling spectroscopy (STS) technique. I already discussed these in Section (VI E 4) arriving at the conclusion that because of the large N UV sensitivity holography is in this regard not a useful navigational aid.

ARPES is just the most direct way to find out whether one is dealing with a Fermi-liquid. The quasiparticles manifest themselves as poles, Lorentzian peaks that disperse around as function of momentum mapping out the band structure. Their widths reflect the imaginary part of the self energy, typically varying like ω^2. In fact, when ARPES started to work in the 1970's it had a great impact directly demonstrating that band structure does exist in a quite literal fashion in simple electron systems.

Because of practical circumstances such as the two dimensional nature of the cuprate electron systems ARPES works particularly well in copper oxides. When in the mid 1990's high resolution data became available these seemed to show a lot of dispersing peaks suggestive of band structure, including "Fermi-surfaces" of the kind expected from the DFT band structure computations. This had a big impact – one heard people say that without ARPES nobody would have gotten the idea to explain "high Tc" in terms of Fermi-liquid style quasiparticles. After the fact the quasiparticles were again occupying the main stage – the fermiology main stream I already alluded to.

Ironically, it is very recently becoming clear that these "moving peaks" may be quite deceitful in this regard. As the example of the CFT fermions (Fig. 4) illustrates, dispersing peaks do not prove anything regarding quasiparticles. Whether one is dealing with quasi-particles or either "un-particles" is encoded in the analytical form of the *line shapes.*

Especially the energy dependence of the ARPES line shapes is for reasons associated with the imperfections of the technique not easy to nail down empirically.

To recognize the best established "unparticle signatures" in the lineshapes one has to focus in on the underdoped regime, while it is pending the direction of the momentum. One distinguishes in the Brillouin zone the directions corresponding with the lattice directions in real space – the "anti-nodal" direction – from the zone diagonal "nodal direction". The d-wave superconducting gap is at maximum at the anti-nodes while it is vanishing at the nodes, explaining the terminology. Independent of doping, rather sharp peaks are found to disperse near the nodes. However, at the anti-nodes there is a lot of action. This refers to the recent detection of a "first order like" transition by ARPES [75]. Above the critical doping one discerns at the anti-nodes also "quasiparticle" peaks for a fixed momentum as function of energy ("energy distribution curves", EDC's). But upon crossing p_c into the underdoped regime these abruptly vanish being replaced by featureless "unparticle" continua. This is the signature of the "first-order like" transition discussed in Section (IX A).

Remarkably, when temperature is decreasing through the superconducting transition sharp peaks appear also at the antinodes, growing from the incoherent spectral weight. Remarkably, the spectral weight of these peaks appears to scale with the *superfluid density* [84]. This is completely beyond anything that one can discern from our understanding of conventional BCS superconductivity. There is no doubt that deep in the superconducting state *genuine* quasiparticles of the Bogoliubov kind are present. The sharp peaks spanning up a whole "Fermi-surface" seen in ARPES are consistent with the phase sensitive "quasi-particle interferences" measured in STS demonstrating directly their quantum mechanical wave nature. But this is in the greater context not surprising: the BCS-like superconducting state is a SRE product and as a vacuum it has to support quasiparticles. These appear to reconstruct in the superconducting state from a normal state where they are absent at least in the underdoped regime.

In turn, these have bearings on another set of observations playing a key role in the belief system of a group who is in the grip of trying to prove that the Hertz-Millis style QCP is the culprit. At very low temperatures and high magnetic fields quantum oscillations are observed suggestive of the existence of a Fermi surface associated with a genuine Fermi-liquid. One deduces from these measurements the kind of mass enhancements expected for such a QCP. The trouble is however that it is established beyond any doubt that these systems are still deep in the "vortex fluid" regime given the magnetic fields that can be achieved. This means that these systems are still fully-fledged superconductors with a relatively low density of free magnetic fluxoids that take care of the dissipative properties of the fluid. Hence, the "Bogoliubons" stabilized by this superconductivity are persisting and although it is still a bit in the dark how these manage to form a Fermi-surface this is obviously not telling anything about what is going on in the normal state at room temperature.

This is the instance to introduce the stunningly strange nature of the actual line shapes in the metallic state. Although noticed early on [83] this is so strange that it is completely overlooked by large parts of the community. Let us focus on the "unparticle" antinodal continua in the underdoped strange metal. Instead of measuring the line shape as function of energy one can also fix energy and vary momentum ("momentum distribution curves", MDC's). Also at the anti-nodes the MDC's correspond with rather sharp Lorentzian peaks, with a width comparable to the nodal fermions, despite the fact that energy wise these are smooth branchcut like continua! This is just very mysterious.

Measuring the MDC linewidths is for technical reasons easier than to study the EDC's. High precision measurements were carried out of the energy dependence of these momentum space widths $\Gamma_k(\omega)$, especially in the nodal regime. At optimal doping $\Gamma_k(\omega) \sim \omega$, the traditional main stay of the marginal Fermi-liquid school of thought: this one obtains assuming free fermions decaying by second order perturbation theory in the "marginal" continuum of Section (IX B 3). This story then continuous claiming that transport and so forth is just like in the Fermi-liquid except that now the single particle life time is set by this inverse life time $\sim \omega$ or T, as usual. One can then continue to blame all the strangeness to be caused by this anomalous decay of the quasiparticles: this is the central assertion of the old marginal Fermi-liquid idea [45].

However, recently it was found that the scaling of this MDC width is actually rather strongly doping dependent (the "nodal liquid") [85]. The results are shown in Fig. (20).Defining $\Gamma_k(\omega) = \Gamma_0 + \lambda \left((\hbar\omega)^2 + (\beta k_B T)^2\right)^{\alpha_p}$ one obtains high quality fits. Γ_0 is a small "elastic" broadening while the remainder shows the E/T scaling. The parameter β relating temperature and energy is $\sim \pi$ and rather doping independent, as is the "coupling" λ. However, the exponent α_p is strongly doping dependent, ranging all the way from 0.3 to 0.7 in the indicated doping range, acquiring the "marginal" value 0.5 at optimal doping. Yet again, this signals smoothly varying anomalous dimensions indicating that one is dealing

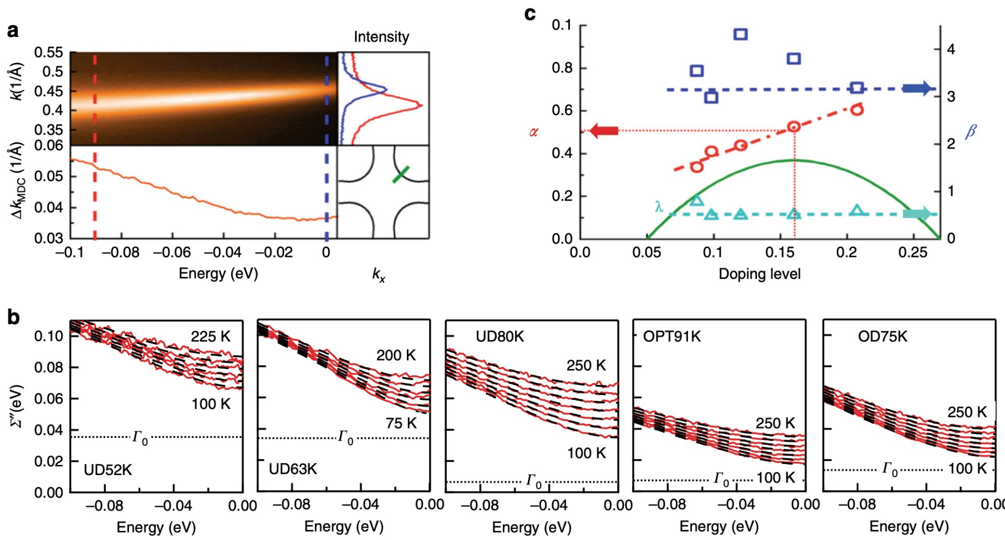

FIG. 20. The measurement of the momentum-width of the "quasiparticle" peaks by angular resolved photoemission [85]. (a) False colour imagine of the spectral function near the nodes as function of momentum and energy, illustrating the extraction of the momentum width Γ_k, see main text. (b) The momentum width as function of doping, showing (c) that only the exponent α_p (red line) is strongly doping dependent.

with a quantum critical *phase*. It is in fact of the same kind as found for the RN strange metal holographic fermions [43] of Section (VI E 4). One is tempted to associate with the "second order perturbation theory" decay of quasiparticles in some kind of quantum critical heat bath. But there is perhaps no better way to understand the deep theoretical inconsistency invoked in such an intuitive assignment as by the reasons discussed in Section (VI E 4), identifying it as a large N pathology.

But let's get back at the EDC's. I already highlighted the strange disconnect between momentum- and energy line shape so manifest for the underdoped antinodal response. But what happens along the nodes? In a recent high precision EDC line shape study near the nodes it was revealed that a similar line shape anomaly is actually at work [86]. As it turns out the nodal EDC's are of the following form,

$$A(k,\omega) = \frac{1}{\pi}\frac{W(k,\omega)\Gamma_k(\omega)}{(\omega - v_F k)^2 + \Gamma_k^2(\omega)}$$
$$W(k,\omega) = A_W + B_W \omega^{\alpha_W} \tag{76}$$

Near the nodes these are characterized by a linear dispersion $\varepsilon_k = v_F k$ while $\Gamma_k(\omega)$ refers to the MDC width that I just discussed. This is just business as usual except for the factor $W(k,\omega)$ in the numerator. As it turns out, this encodes for a "fat" powerlaw tail ($\alpha_W > 0$). This appears to be at maximum around optimal doping, gradually diminishing until $W \simeq 1$ in the strongly overdoped regime.

The take home message is that the "branchcut EDC/quasiparticle pole MDC" that is so obvious at the antinodes in the underdoped metallic regime is actually still at work even in the nodal regime up to high dopings. Given the difficulties to determine the EDC lineshapes this was just hidden from view until very recently. Once again, this frequency dependence is reminiscent of the "unparticle" branch cut spectra of CFT's (e.g., Fig. 4). However, their MDC's are also of the branch cut form while here the momentum dependence is quasiparticle like. Although there is not enough data available for a mathematical proof – for Kramers-Kronig reasons the spectral functions above E_F are required – one can make the case that it is virtually impossible to associate this with an outcome for a perturbative self-energy controlled by the free limit.

It is presently completely unclear how to discern the underlying physical principle that is responsible for this momentum wise quasiparticle – energy wise unparticle behaviour. So much is clear that given the former one cannot possibly jump to the presently widespread community belief largely based on ARPES that the cuprate strange metals are in essence Fermi-liquids with some special effects. To be continued.

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