Exam Quantum Theory, 13 January 2025, 9–12 hours.

1. The edge state in the quantum spin Hall effect consists of a spin-up wave function $\psi_\uparrow(x)$ and a spin-down wave function $\psi_\downarrow(x)$, extended along the x-axis. At energy E these states satisfy the differential equations

$$V(x)\psi_\uparrow(x) - i\hbar v\frac{d}{dx}\psi_\uparrow(x) = E\psi_\uparrow(x),$$
$$V(x)\psi_\downarrow(x) + i\hbar v\frac{d}{dx}\psi_\downarrow(x) = E\psi_\downarrow(x).$$

where $V(x)$ is the electrical potential on the edge and v is a constant velocity.

- *a)* Explain why these two equations satisfy the condition of time-reversal symmetry.
- *b)* Verify that the energy spectrum satisfies Kramers degeneracy.
- *c)* Show that the electron density $\rho(x) = |\psi_\uparrow(x)|^2 + |\psi_\downarrow(x)|^2$ along the edge is uniform, independent of x.

2. The coherent state of a harmonic oscillator is given by

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle,$$

for any complex number $\alpha \in \mathbb{C}$, where $|n\rangle$ is the state with occupation number n.

- *a)* The number state $|n\rangle$ evolves in time as $|n(t)\rangle = e^{-in\omega t}|n(0)\rangle$, with ω the oscillator frequency. Show that a coherent state at time $t = 0$ remains a coherent state at later times.
- *b)* Calculate the overlap $\langle\alpha_1|\alpha_2\rangle$ of two coherent states; are they orthogonal for $\alpha_1 \neq \alpha_2$?
- *c)* Consider the operator $|\alpha\rangle\langle\alpha|$ and integrate α over the complex plane,

$$I = \frac{1}{\pi}\int_{\mathbb{C}} d\alpha\,|\alpha\rangle\langle\alpha|.$$

Prove that I is the identity operator.*

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*You may use the integral formula $\int_{\mathbb{C}} d\alpha\, e^{-|\alpha|^2}(\alpha^*)^n\alpha^m = \pi m!\delta_{nm}$.

3. We consider the vacuum electromagnetic energy E inside a single-mode wave guide, of length L, closed at the two ends by metal boundaries. The wave vector k has only components along the wave guide, equal to $k = \pi n/L$, with $n = 1, 2, 3, \ldots$. The vacuum energy contribution from each wave vector (speed of light c) is $\frac{1}{2}\hbar c k e^{-k/k_c}$. The exponential factor enters because waves of wave number $k \gtrsim k_c$ are suppressed by the resistivity of the metal boundaries.

- *a)* Show that for large k_c the vacuum energy has the Taylor expansion†

$$E(L) = \tfrac{1}{2}\pi\hbar c \left(\frac{Lk_c^2}{\pi^2} - \frac{1}{12L} + \text{order}(1/k_c^2) \right).$$

- *b)* We insert a metal plate in the wave guide, as shown in the figure, at a distance a from one end and at a distance b from the other end. What is now the vacuum energy of the entire system for large k_c?
- *c)* Calculate the force on the metal plate when $b \gg a$. In which direction does it point?

4. The Hamiltonian of electrons in graphene is a 2×2 matrix,

$$\hat{H} = \begin{pmatrix} 0 & v(\hat{p}_x - i\hat{p}_y) \\ v(\hat{p}_x + i\hat{p}_y) & 0 \end{pmatrix} \tag{1}$$

where v is a constant velocity and $\hat{p}_x, \hat{p}_y$ are the two components of the momentum operator in the x-y plane. (There is no motion in the z-direction.)

- *a)* Calculate the energy spectrum $E(p_x, p_y)$ of graphene. Is there a lowest energy? *Hint: First calculate $\hat{H}^2$.*

In the presence of a uniform magnetic field B in the z-direction, the Hamiltonian of graphene is modified by the substitution $p_y \mapsto p_y - eBx$. The energy spectrum now consists of Landau levels.

- *b)* Show that there exists a B-independent Landau level at energy $E = 0$. *Hint: See if you can construct a zero-energy wave function of either the form*

$$\psi_1(x, y) = \begin{pmatrix} 0 \\ e^{iky} f(x) \end{pmatrix} \text{ or of the form } \psi_2(x, y) = \begin{pmatrix} e^{iky} f(x) \\ 0 \end{pmatrix},$$

for some constant k and some function $f(x)$.

- *c)* The classical motion of an electron in a magnetic field is a cyclotron orbit and the Landau level then follows from the quantization of this periodic motion. Explain the existence of an $E = 0$ Landau level in graphene from this semiclassical point of view.

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†You may use that $\sum_{n=1}^{\infty} n e^{-\alpha n} = 1/\alpha^2 - 1/12 + \text{order}(\alpha^2)$.