Answers to the Exam Quantum Theory, 4 March 2025

each item gives 2 points for a fully correct answer, grade = total $\times\, 9/24 + 1$

1. *(a)* Yes, it still holds that $\langle\psi|H(t)\psi'\rangle = \langle H(t)\psi|\psi'\rangle$, the time dependence of the potential has no effect on the Hermitian conjugate.
   (b) $[H(t_1), H(t_2)] = [-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}, \frac{1}{2}m(\omega^2(t_2) - \omega^2(t_1))x^2] \neq 0$,
   because $[d^2/dx^2, x^2] = 2 + 2x d/dx \neq 0$.
   (c) $dE/dt = \langle\psi|dH/dt|\psi\rangle + \langle\psi|H|(i\hbar)^{-1}H\psi\rangle + \langle(i\hbar)^{-1}H\psi|H|\psi\rangle$
   $= \langle\psi|dH/dt|\psi\rangle = m\omega(d\omega/dt)\langle\psi|x^2|\psi\rangle$.

2. *(a)* $[c, c^\dagger] = [a\cosh\lambda + b^\dagger\sinh\lambda, a^\dagger\cosh\lambda + b\sinh\lambda] = \cosh^2\lambda - \sinh^2\lambda = 1$, similarly $[d, d^\dagger] = 1$. The other commutators vanish, in particular, $[c, d] = ([a, a^\dagger] + [b^\dagger, b])\cosh\lambda\sinh\lambda = 0$.
   (b) Substituting the expressions for c and d gives $c^\dagger c + dd^\dagger = (\cosh^2\lambda + \sinh^2\lambda)(a^\dagger a + bb^\dagger) + (2\cosh\lambda\sinh\lambda)(a^\dagger b^\dagger + ab)$; note that $\cosh^2\lambda + \sinh^2\lambda = \cosh 2\lambda$ and $2\cosh\lambda\sinh\lambda = \sinh 2\lambda = \gamma\cosh 2\lambda$, so that $(\cosh 2\lambda)^{-1}(c^\dagger c + dd^\dagger) = a^\dagger a + bb^\dagger + \gamma(a^\dagger b^\dagger + ab)$.
   (c) The operators $c^\dagger c$ and $dd^\dagger = d^\dagger d + 1$ have eigenvalues $0, 1, 2, \ldots$ and $1, 2, 3, \ldots$, respectively, so the spectrum of H consists of integer multiples of $(\cosh 2\lambda)^{-1} = \sqrt{1-\gamma^2}$.

3. *a)* $\langle\psi|\psi\rangle = \sum_{n,m} c_n^* c_m \langle\Phi_n|\Phi_m\rangle = \sum_n |c_n|^2$, because $\langle\Phi_n|\Phi_m\rangle = \delta_{nm}$.
   b) $\langle\psi|H - E_0|\psi\rangle = \sum_{n,m} c_n^* c_m \langle\Phi_n|H - E_0|\Phi_m\rangle = \sum_{n,m} c_n^* c_m (E_m - E_0)\langle\Phi_n|\Phi_m\rangle = \sum_n |c_n|^2(E_n - E_0) \geq 0$.
   c) Calculate $E(a) = \int_{-\infty}^{\infty} \Phi_a^*(x) H \Phi_a(x) dx$, and solve $dE(a)/da = 0$ to find the minimal value E_{min} of $E(a)$ as a function of $a > 0$. This is the optimal upper bound of the ground state energy E_0.

4. *a)* $d\vec{r}/dt = (i/\hbar)[H, \vec{r}] = (1/m)(\vec{p} - q\vec{A})$
   b) $\exp(iq\chi/\hbar)(\vec{p} - q\vec{A})^2\exp(-iq\chi/\hbar) = [\exp(iq\chi/\hbar)(\vec{p} - q\vec{A})\exp(-iq\chi/\hbar)]^2 = (\vec{p} - q\vec{A} - q\nabla\chi)^2 = (\vec{p} - q\vec{A}')^2$
   c) H' is related to H by a unitary transformation, so it represents the same system described in a different basis.