Answers to the Exam Quantum Theory, 13 January 2025

each item gives 2 points for a fully correct answer, grade = total $\times 9/24 + 1$

1. *(a)* Time reversal symmetry means that if $(\psi_\uparrow, \psi_\downarrow)$ is a solution, then also $(\psi_\downarrow^*, \psi_\uparrow^*)$. The equations satisfy that symmetry (take complex conjugation and switch spin-up and spin-down).
   (b) For each energy E there are two independent solutions, one with spin-up and one with spin-down, so the spectrum has the two-fold Kramers degeneracy.
   (c) The solution is $\psi_\uparrow(x) = \exp\left(\frac{i}{\hbar v}\int_0^x [E - V(x')]\,dx'\right)\psi_\uparrow(0)$,
   $\psi_\downarrow(x) = \exp\left(-\frac{i}{\hbar v}\int_0^x [E - V(x')]\,dx'\right)\psi_\downarrow(0)$, so $\rho(x) = \rho(0)$.

2. *(a)* The state $|\alpha\rangle$ evolves in time as $e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^\infty \frac{\alpha^n}{\sqrt{n!}}e^{-in\omega t}|n(0)\rangle$, which is a coherent state with a time dependent $\alpha(t) = e^{-i\omega t}\alpha(0)$.
   (b) Use $\langle n|m\rangle = \delta_{nm}$ to obtain $\langle\alpha_1|\alpha_2\rangle = e^{-|\alpha_1|^2/2}e^{-|\alpha_2|^2/2}\sum_{n=0}^\infty \frac{(\alpha_1^*\alpha_2)^n}{n!} = e^{\alpha_1^*\alpha_2}e^{-|\alpha_1|^2/2}e^{-|\alpha_2|^2/2}$; this is nonzero for $\alpha_1 \neq \alpha_2$, so coherent states are not orthogonal.
   (c) Using the given integral formula, you find that $I = \sum_{n=0}^\infty |n\rangle\langle n|$ which is a resolution of the identity.

3. *a)*
   $$E(L) = \frac{\pi\hbar c}{2L}\sum_{n=1}^\infty n e^{-\alpha n}, \text{ with } \alpha = \frac{\pi}{Lk_c},$$
   and then substitute $\sum_{n=1}^\infty n e^{-\alpha n} = 1/\alpha^2 - 1/12 + \text{order}(\alpha^2)$.
   b) $E_{\text{tot}}(a,b) = E(a) + E(b) = \frac{1}{2}\pi\hbar c\big((a+b)k_c^2/\pi^2 - 1/12a - 1/12b\big)$
   c) Vary a at fixed $L = a + b$, so $b = L - a$;
   $F = -dE_{\text{tot}}(a, L-a)/da = -\frac{1}{2}\pi\hbar c(1/12a^2 - 1/12(L-a)^2) \to -\pi\hbar c/24a^2$.
   The force points to the right in the figure.

4. *(a)* $H^2 = v^2(p_x^2 + p_y^2)$ times the unit matrix, so $E^2 = v^2(p_x^2 + p_y^2)$; there are positive and negative energies, without a lowest energy.
   (b) first choice:
   $$H\psi_1 = i(eBx - \hbar k)f(x) - i\hbar f'(x) = 0 \Rightarrow f(x) = \exp(eBx^2/2\hbar - kx)f(0)$$
   — fails because it is not normalizable; second choice
   $$H\psi_2 = -i(eBx - \hbar k)f(x) - i\hbar f'(x) = 0 \Rightarrow f(x) = \exp(-eBx^2/2\hbar + kx)f(0)$$
   is normalizable, so we have found our zero-energy eigenfunction.
   (c) At zero energy the only phase shift accumulated along a periodic orbit is at the turning points, twice $-\pi/2$, plus the Berry phase of π from the circulating spin in graphene, so the net phase shift is zero, hence there is a bound state at zero energy.