Problem set for the
Introduction to linear algebra for quantum information

based on ***Quantum Computation and Quantum Information****,*
by Nielsen & Chuang, §2.1

1. Prove the Cauchy-Schwarz inequality

$$\langle v|v\rangle\langle w|w\rangle \geq |\langle v|w\rangle|^2.$$

   Hint: Insert the resolution of the identity, $\langle v|v\rangle = \sum_i \langle v|i\rangle\langle i|v\rangle$, and ask yourself what would happen if you would keep only a single term in the sum.

2. Write the Pauli operators as an outer product of the basis vectors $|0\rangle$ and $|1\rangle$.

3. Consider the operator $V = |v\rangle\langle v|$ for some given nonzero vector $|v\rangle$. Prove that V has one eigenvalue equal to 1, while all other eigenvalues are equal to 0.

4. Is the product of two Hermitian operators Hermitian? Is the product of two unitary operators unitary?

5. Calculate the matrix representation of the tensor products $X \otimes I$ and $X \otimes Y$.

6. Calculate the matrix representation of the Hadamard operator

$$H = \frac{1}{\sqrt{2}}\left[(|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 1|\right].$$

7. Calculate eigenvalues and eigenvectors for the Pauli matrices X, Y, Z. Are these operators Hermitian? Are they unitary?

8. Prove that the matrix $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$ is not diagonalizable. What are its singular values?

9. An operator of the form

$$P = \sum_{i=1}^{k} |i\rangle\langle i|,$$

   constructed by choosing k basis vectors out of an orthonormal set of n basis vectors is called a *projector*. If $k = n$ then $P = I$, but if $k < n$ this is not the case
   Prove that $P^2 = P$. What are its eigenvalues?

10. A unitary transformation of an operator A is defined by

$$A' = UAU^\dagger,$$

    with U a given unitary operator.
    Prove that if λ is an eigenvalue of A, then λ is also an eigenvalue of A'.

11. An operator is called *positive* if $\langle v|A|v\rangle$ is real and ≥ 0 for any vector $|v\rangle$. For example, $BB^\dagger$ is a positive operator for any B.
    Prove that the set of positive operators is the set of Hermitian operators with nonnegative eigenvalues.

12. Prove that $\mathrm{Tr}\, AB = \mathrm{Tr}\, BA$. Also prove that, for any operator A and any vector v,

$$\mathrm{Tr}\, A|v\rangle\langle v| = \langle v|A|v\rangle.$$

13. A function $f(A)$ of a diagonalizable operator $A = \sum_a a|a\rangle\langle a|$ is defined by

$$f(A) = \sum_a f(a)|a\rangle\langle a|.$$

Prove that

$$\exp(i\theta\sigma_x) = I\cos\theta + i\sigma_x \sin\theta.$$

Hint: Expand the exponent in a Taylor series and consider separately the even and the odd powers of θ.
More generally, if $\Sigma = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$ with $\vec{n}$ a unit vector, then

$$\exp(i\theta\Sigma) = I\cos\theta + i\Sigma\sin\theta.$$