Quantum Information: lecture 2

- entanglement — Preskill 2.4
- teleportation — Preskill 4
- Bell inequality

in this lecture, an overview of

measures of entanglement for pure states (concurrence, entanglement entropy), Bell inequality, no cloning theorem, teleportation

EPR paradox & entangled states

Einstein, Podolsky & Rosen (1935): “spooky action at a distance”

$$\Psi = \frac{1}{\sqrt{2}}|\uparrow\rangle_A|\downarrow\rangle_B - \frac{1}{\sqrt{2}}|\downarrow\rangle_A|\uparrow\rangle_B$$

when B measures $\downarrow$ the state of A collapses “instantaneously” to $\uparrow$

actually, there is no violation of the principle of relativity, as follows by comparing the reduced density matrix $\rho_A = \mathrm{Tr}_B\,\rho$ before and after the measurement: in both cases the answer is the same,

$$\rho_A = \tfrac{1}{2}|\uparrow\rangle\langle\uparrow| + \tfrac{1}{2}|\downarrow\rangle\langle\downarrow|$$

$\rho_A^2 \neq \rho_A$, so A is in a mixed state, while A en B together were in a pure state: A and B are entangled

product state $|\uparrow\rangle_A|\downarrow\rangle_B \Rightarrow \rho_A = |\uparrow\rangle\langle\uparrow|$ remains pure.

entanglement measures

must be invariant under local unitary transformations

two-level system *in a pure state*:

$$\Psi = \alpha|\uparrow\rangle_A|\downarrow\rangle_B + \beta|\downarrow\rangle_A|\uparrow\rangle_B, \quad |\alpha|^2 + |\beta|^2 = 1.$$

the product $C = 2|\alpha\beta|$ (the concurrence) measures how far Ψ is from a product state: $C = 0$ for a product state and $C = 1$ for a maximally entangled state.

more generally, for $\Psi = \sum_{nm} c_{nm}|n\rangle_A|m\rangle_B$ one has $C = 2|\det c|$ — see exercise 1

2^N levels: entanglement entropy

$$S = -\mathrm{Tr}\,\rho_A{}^2\log\rho_A = -\mathrm{Tr}\,\rho_B{}^2\log\rho_B$$

varies between 0 (ρ_A is pure) and N ($\rho_A = 2^{-N}\times$ unit matrix)

$\rho_A \mapsto cc^\dagger$ and $\rho_B \mapsto c^T c^*$ have the same eigenvalues ρ_n and thus the same $S = -\sum_n \rho_n \log \rho_n$

Bell inequality

the spins of A and B in the entangled state
$\Psi = \frac{1}{\sqrt{2}} |\uparrow\rangle_A |\downarrow\rangle_B - \frac{1}{\sqrt{2}} |\downarrow\rangle_A |\uparrow\rangle_B$ are *anticorrelated in any basis,*

$$\langle \hat{a}\hat{b} \rangle \equiv \langle \Psi | (\hat{a} \cdot \sigma_A)(\hat{b} \cdot \sigma_B) | \Psi \rangle = -\hat{a} \cdot \hat{b} = -\cos\theta$$

John Bell (1964): method to distinguish classical from quantum correlations, by comparing correlations in different basis.

take four coplanar unit vectors $\hat{a}'$, $\hat{b}$, $\hat{a}$, $\hat{b}'$ separated by 45°:

$\langle \hat{a}\hat{b} \rangle = \langle \hat{a}'\hat{b} \rangle = \langle \hat{a}\hat{b}' \rangle = -\cos\pi/4 = -1/\sqrt{2}$, while $\langle \hat{a}'\hat{b}' \rangle = -\cos 3\pi/4 = 1/\sqrt{2}$, so

$\mathcal{B} \equiv \langle \hat{a}\hat{b} \rangle + \langle \hat{a}'\hat{b} \rangle + \langle \hat{a}\hat{b}' \rangle - \langle \hat{a}'\hat{b}' \rangle = -2\sqrt{2}$.

But classically, $|\mathcal{B}| \leqslant 2$, because if $A, B, A', B' \in \{\pm 1\}$, then $AB + A'B + AB' - A'B' = A(B + B') + A'(B - B') = \pm 2$, so the average is between -2 and $+2$.

this Bell inequality is tested experimentally with photon polarizations, eliminating "hidden variable theories".

No-cloning theorem

it is not possible to copy an arbitrary unknown quantum state

Dieks, Wootters & Zurek (1982)

there exists no unitary operator U such that for any pure state $|\phi\rangle$

$$U|\phi\rangle_A|0\rangle_B = e^{i\alpha(\phi)}|\phi\rangle_A|\phi\rangle_B$$

Proof:

$$\begin{aligned}\langle\phi|\psi\rangle &= \langle 0|_B\langle\phi|_A|\psi\rangle_A|0\rangle_B = \langle 0|_B\langle\phi|_A U^\dagger U|\psi\rangle_A|0\rangle_B \\ &= e^{i(\alpha(\psi)-\alpha(\phi))}\langle\phi|_B\langle\phi|_A|\psi\rangle_A|\psi\rangle_B \\ &= e^{i(\alpha(\psi)-\alpha(\phi))}\langle\phi|\psi\rangle^2\end{aligned}$$

$$\Rightarrow |\langle\phi|\psi\rangle| = |\langle\phi|\psi\rangle|^2 \Rightarrow |\langle\phi|\psi\rangle| = 0 \text{ or } 1.$$

This can not be the case for two arbitrary states.

Quantum teleportation

2 classical bits can transmit the unknown state of 1 qubit, provided sender and receiver share an entangled qubit pair

Bennett, Brassard, Crépeau, Jozsa, Peres & Wootters (1993)

- *no-cloning theorem OK*
- *special relativity OK*

It's teleportation Jim, but not as we know it.