ANSWERS TO THE EXAM QUANTUM INFORMATION, 15 DECEMBER 2020

each item gives 2 points for a fully correct answer, grade = total $\times 9/24 + 1$

1. *a)* $\text{CNOT}|\Psi\rangle = \sqrt{\frac{1}{3}}|0\rangle|0\rangle + \sqrt{\frac{2}{3}}|1\rangle|0\rangle$, then applying a Hadamard on the first qubit gives $\sqrt{\frac{1}{6}}(|0\rangle + |1\rangle)|0\rangle + \sqrt{\frac{1}{3}}(|0\rangle - |1\rangle)|0\rangle = c_+|0\rangle|0\rangle + c_-|1\rangle|0\rangle$, with $c_\pm = \sqrt{\frac{1}{6}} \pm \sqrt{\frac{1}{3}}$.
   b) concurrence is 0, the qubits are not entangled.
   c) the final state is a product state of $|0\rangle$ for the second qubit and $|\psi_A\rangle = c_+|0\rangle + c_-|1\rangle$ for the first qubit, so the reduced density matrix of the first qubit (with Alice) is $\rho = |\psi_A\rangle\langle\psi_A|$. This is a pure state.

2. *(a)*

$$\frac{d}{dt}\text{Tr}\,\rho(t) = \frac{1}{i\hbar}\text{Tr}\,[H,\rho(t)] = 0,$$

   because $\text{Tr}\,AB = \text{Tr}\,BA$.
   (b) Define $F(t) = \rho^2(t) - \rho(t)$, then calculate

$$i\hbar\frac{\partial F}{\partial t} = \rho[H,\rho] + [H,\rho]\rho - [H,\rho] = [H,F],$$

   so $F(t) = e^{-iHt/\hbar}F(0)e^{iHt/\hbar}$, and since $F(0) = 0$ it follows that $F(t) = 0$.
   (c) $\rho = |\psi\rangle\langle\psi|$, $\rho\psi = \langle\psi|\psi\rangle\psi = \psi$.

3. *(a)* Act on the qubit with a Hadamard gate and measure it. The Hadamard transforms $|\psi_1\rangle \mapsto |0\rangle$ and $|\psi_2\rangle \mapsto |1\rangle$, so the measurement will reveal the state.
   (b) Only orthogonal states can be distinguished with certainty. The two states $|0\rangle$ and $\sqrt{\frac{1}{2}}|0\rangle - \sqrt{\frac{1}{2}}|1\rangle$ are not orthogonal, and since a unitary operation conserves the angle between states, they will remain non-orthogonal no matter how we operate on the qubit. So Bob cannot tell with certainty which qubit he has.
   (c) A unitary operator is invertible, the inverse of the no-deleting statement is the no-cloning theorem:

$$U^\dagger|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle.$$

   So if such a U would exist, we would also be able to clone an arbitrary state using the operator $U^\dagger$, which is forbidden.

4. *(a)* $\frac{1}{2}|0\rangle(|\phi\rangle|\psi\rangle + |\psi\rangle|\phi\rangle) + \frac{1}{2}|1\rangle)(|\phi\rangle|\psi\rangle - |\psi\rangle|\phi\rangle)$
   (b) $\frac{1}{2}(1 + |\langle\phi|\psi\rangle|^2)$
   (c) if $|\phi\rangle = |\psi\rangle$ the probability to measure the state $|1\rangle$ in the control qubit is zero, so if you do measure $|1\rangle$ the states $|\phi\rangle$ and $|\psi\rangle$ must have been orthogonal.