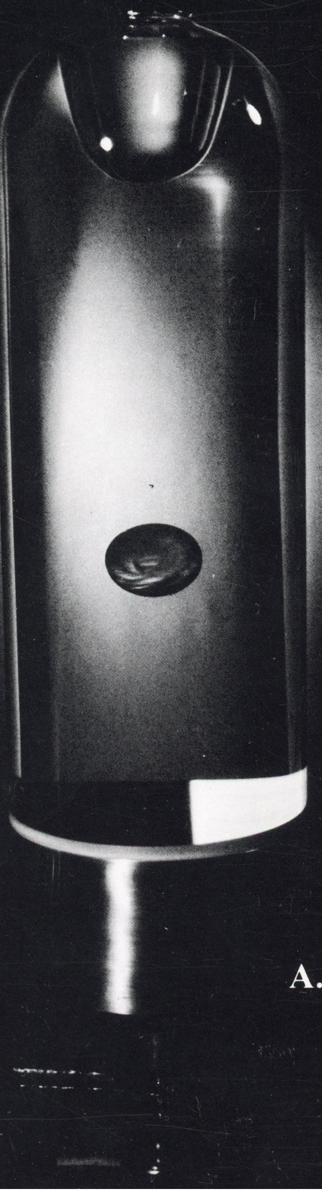
DRAG ON A SPHERE MOVING SLOWLY ALONG THE AXIS OF A ROTATING VISCOUS FLUID



A.J. WEISENBORN

Vlast dissertaties

DRAG ON A SPHERE MOVING SLOWLY ALONG THE AXIS OF A ROTATING VISCOUS FLUID

PROEFSCHRIFT

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Overige leden:	Prof. dr. F.A. Berends
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The cover photograph shows a sphere rising
in a rotating vessel filled with water.

Leden van de promotiecommissie:

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aan mijn moeder, vader,
zussen en broers.

CONTENTS

PREFACE

The aim of the research presented in this thesis is the evaluation of the drag on a sphere moving slowly along the axis of a rotating fluid. This drag has up to now only been calculated for very small and very large values of the Taylor number, the dimensionless measure for the rotational velocity of the fluid.

The first calculation of the drag on a sphere moving in a rotating fluid was performed by Grace [1]) in 1926. He evaluated the ultimate drag on a sphere, impulsively set in motion in a rotating ideal fluid. Grace's expression for the drag contained an approximate numerical coefficient. In 1952 Stewartson [2]) calculated the exact value of this coefficient. He also gave an analytic expression for the velocity field in the fluid and in particular for the "Taylor column" produced by the motion of the sphere. This phenomenon had been observed experimentally many years earlier, in 1922, by Taylor [3]).

In 1969 Moore and Saffman [4]) have analysed the structure of the velocity field caused by the motion of a sphere along the axis of a rotating viscous fluid at large values of the Taylor number. They neglected all momentum convection in the rotating frame of reference. This means that they considered the fluid motion in the limit of zero Reynolds number (the Reynolds number R is the dimensionless measure for the translational velocities in the fluid). They showed that the drag on the sphere becomes for large values of T asymptotically equal to Stewartson's value.

In 1970 Maxworthy [5]) has measured the drag for large Taylor numbers and Reynolds numbers above 1. Extrapolation of his measurements to zero Reynolds number yielded an asymptotic value for the drag which exceeds Stewartson's value by approximately 50%. Up to the present day no satisfying explanation has been given for this rather large difference between theory and experiment.

In 1964 Childress [6]) has studied the drag in the regime in which both the Taylor number and the Reynolds number are very small, viz. $T \ll 1$ and $R \ll 1$. He was able to determine a first correction to the drag at $T = 0$ and $R = 0$, proportional to $T^{\frac{1}{2}}$. For very small values of T and R Childress' result for the drag is in good agreement with Maxworthy's 1965 measurements [7]) of this quantity.

All results for the drag quoted above were obtained by means of the conventional method for calculating this quantity. This method requires explicit knowledge of the solution of the equation of motion for the fluid. The calculations of the velocity field at very small and very large Taylor numbers (see refs. 6 and 4, resp.) show that already in these limits the structure of the velocity field is very complicated. It seems reasonable to expect that this structure will be even more complicated for intermediate values of T. It is therefore not surprising that the drag has up to now not been computed for these values of T.

In this dissertation we develop an alternative method which will enable us to calculate systematically successive approximations to the drag for <u>all</u> values of the Taylor number. This method, which allows us to evaluate this quantity without any explicit knowledge of the solution of the equation of motion, makes use of the concept of induced forces. This concept was proposed by Oseen [8]) and used extensively by Burgers [9]). With the help of this concept Mazur and Bedeaux [10]) developed a exact analytic method which enabled them to obtain a generalization of Faxén's first theorem* to finite frequencies. Using an extended version of this method Mazur and Van Saarloos [11]) were able to analyse hydrodynamic interactions between an arbitrary number of spheres. The method developed in this thesis is based on this extended version and rests on the introduction of an induced force density in the equation of motion for the fluid. The velocity field may then be solved in terms of this force density. Subsequently this force density is expanded in irreducible force multipoles. By appropriate use of the boundary conditions at the surface of the sphere a hierarchy of equations may then be derived for these force multipoles. The drag force, the first force multipole, may be solved from this hierarchy of equations by elimination of all higher multipoles.

To demonstrate the usefulness of the method we first apply it to two classic hydrodynamic problems. In chapter I we evaluate the drag on an infinitely long circular cylinder, at rest in a perpendicular uniform stationary flow; in chapter II we analyse the drag on a sphere, at rest in a similar flow. In both cases we describe the motion of the fluid with the Oseen equation. This (linear) equation is a good approximation to the full Navier-Stokes equation for small values of the Reynolds number, i.e. for very low flow velocities. The results of the calculations therefore only have physical

* Faxén's first theorem gives the relation between the force exerted on a sphere and the unperturbed velocity field in the case that all momentum convection is neglected.

significance in this regime. On the other hand the results of the present analyses may be compared to numerical solutions of the Oseen equation. Such a comparison shows that the approximation to the drag based on the first two multipoles alone yields already very satisfactory values for this quantity for Reynolds numbers of order 10 .

In chapter III we then study the drag on a sphere moving slowly along the axis of a rotating fluid. We neglect all momentum convection in the rotating frame of reference, i.e. we consider the fluid motion in the limit of zero Reynolds number. For small values of the Taylor number we expand the drag in a power series in $T^{\frac{1}{2}}$ and evaluate explicitly seven terms of this series. Childress' correction [6]) is recovered as first term. We further show that the approximation to the drag, based on the first three force multipoles, becomes for large values of the Taylor number asymptotically equal to Stewartson's result. Finally we evaluate for all Taylor numbers the approximations to the drag, based on the first, first three and first five force multipoles. The results show that the difference between the last two approximations is less than 1% for all values of T. In chapter IV we analyse the influence of convection on the drag studied in chapter III. For this purpose we take into account momentum convection in Oseen's approximation.

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Chapter I

The Oseen drag on a circular cylinder revisited

This chapter has appeared as a paper in Physica 123A (1984) 191-208.

THE OSEEN DRAG ON A CIRCULAR CYLINDER REVISITED

1. Introduction

The motion of an infinitely long circular cylinder through a viscous unbounded fluid in a direction perpendicular to its axis is a classic problem, which combines the apparent simplicity of two-dimensional hydrodynamics with its underlying difficulty.

The first treatment of this problem was given in 1851 by Stokes[1]), who noticed that the fully linearized Navier–Stokes equation – commonly referred to as the Stokes equation – does not have a solution for this case (Stokes paradox). In 1911 Oseen[2]) introduced an "extended Stokes equation", now known as the Oseen equation, to remedy some of the difficulties and paradoxes inherent in the Stokes equation. On the basis of this equation Lamb[3]) found an approximate solution for the drag, or equivalently the mobility of a cylinder at low velocities, or, to be more precise, Reynolds numbers. Later on two other solutions for the Oseen drag were proposed, by Bairstow et al.[4]) and by Faxén[5]). Bairstow's solution does not lead to a substantial improvement of Lamb's result for very small Reynolds numbers. At higher Reynolds numbers this solution leads to more reasonable values for the drag than Lamb's, but does not yield values which are quantitatively satisfactory. Faxén's solution on the other hand is not available in a form which enables one to obtain numerical values for the drag. The above treatments have as common feature that first explicit solutions for the velocity and pressure fields are constructed on the basis of the Oseen equation, and that these are then used to calculate the drag by integration of the pressure tensor over the surface of the cylinder.

In this paper we propose an alternative approach in which explicit knowledge of the velocity and pressure fields is not required. Our approach is based on a method of induced forces which was previously used to generalize Faxén's first theorem[6,7]). Recently the method was also applied by Mazur and Van Saarloos[8]) to the problem of many sphere hydrodynamic interactions in Stokes flow. For this problem use was made of an expansion of the induced forces in irreducible force multipoles. The advantage of the method of induced forces referred to, is that the desired results follow directly from an appropriate use of the boundary conditions at the surface of the moving object or objects.

Section 2 will be devoted to the formulation of the problem; we give the formal solution for the velocity field in wavevector representation on the basis of the Oseen equation. In section 3 we obtain a simple expression for the mobility per unit of length by making a simplifying assumption on the nature of the induced force density. In section 4 we give the expansion of the induced force density in terms of irreducible force multipoles. With the aid of this expansion we derive a hierarchy of equations for these multipoles. This hierarchy may then be used to obtain an exact expression for the mobility in the form of an infinite sum by elimination of all irreducible force multipoles in favour of the force itself. The various terms in this sum (or partial sums thereof) correspond to retaining higher and higher multipoles in the expansion of the induced force density. The first term of the series is identical with the simple expression found in section 3. We also give the explicit form of the second term as quotient of integrals over standard functions. In section 5 we compare our results in first and second approximation (i.e. retaining only the zeroth, respectively, the zeroth and first multipole) to the results obtained previously in closed form and also to the results obtained by solving the Oseen equation numerically[9]). From our analysis it follows that taking into account the zeroth and first multipoles alone leads already to satisfactory results for the drag over a large range of values of the Reynolds number.

2. Formulation of the problem

We consider an infinitely long circular cylinder with radius a, at rest in a viscous incompressible fluid. The fluid velocity at infinity, U, is taken constant and perpendicular to the axis of the cylinder. We choose cylindrical coordinates (r, ϕ, z), with the z-axis along the axis of the cylinder. The motion of the fluid is governed by the Oseen equation

$$\left.\begin{aligned} &\rho \boldsymbol{U}\cdot\nabla \boldsymbol{v}(\boldsymbol{r}) + \nabla\cdot \boldsymbol{P}(\boldsymbol{r}) = 0 && (2.1)\\ &\nabla\cdot\boldsymbol{v}(\boldsymbol{r}) = 0 && (2.2)\end{aligned}\right\}\quad \text{for } r > a$$

with

$$P_{\alpha\beta} = p\delta_{\alpha\beta} - \eta\left(\frac{\partial v_\alpha}{\partial r_\beta} + \frac{\partial v_\beta}{\partial r_\alpha}\right). \tag{2.3}$$

Here $\boldsymbol{v}(\boldsymbol{r})$ is the velocity field, $\boldsymbol{P}(\boldsymbol{r})$ the pressure tensor, $p(\boldsymbol{r})$ the hydrostatic pressure, and η and ρ the viscosity and density of the fluid, respectively.

In the above equations $\boldsymbol{r}$ denotes the two-dimensional vector

$$\boldsymbol{r} = (r\cos\phi, r\sin\phi)\,. \tag{2.4}$$

We also introduce $\hat{\boldsymbol{r}}$, the unit vector normal to the surface and pointing in the outward direction; $\hat{\boldsymbol{r}}$ is defined as $\hat{\boldsymbol{r}} \equiv \boldsymbol{r}/r$. Due to translational invariance in the z-direction, field quantities as e.g. $\boldsymbol{v}$ are functions only of the vector $\boldsymbol{r}$ (cf. eqs. (2.1)–(2.3)). We supplement these equations with stick boundary conditions at the surface of the cylinder:

$$\boldsymbol{v}(\boldsymbol{r}) = 0 \quad \text{for } r = a\,. \tag{2.5}$$

Within the context of the method of induced forces, the above set of equations may be replaced by an equivalent one in which the fluid equations are extended within the cylinder and written in the form

$$\rho\boldsymbol{U}\cdot\nabla\boldsymbol{v}(\boldsymbol{r}) + \nabla\cdot\boldsymbol{P}(\boldsymbol{r}) = \boldsymbol{F}_{\mathrm{ind}}(\boldsymbol{r}) \quad \text{for all } r \tag{2.6}$$

$$\nabla\cdot\boldsymbol{v}(\boldsymbol{r}) = 0 \tag{2.7}$$

with $\boldsymbol{F}_{\mathrm{ind}}(\boldsymbol{r}) \equiv 0$ for $r > a$. The extension of the fluid velocity field is chosen as

$$\boldsymbol{v}(\boldsymbol{r}) = 0 \quad \text{for } r \leqslant a \tag{2.8}$$

while the condition

$$p(\boldsymbol{r}) = 0 \quad \text{for } r < a \tag{2.9}$$

is imposed on the hydrostatic pressure.

The above formulation of the problem by means of an induced force density is obviously equivalent with the original boundary value problem. From substitution of eqs. (2.8) and (2.9) into eq. (2.6) it follows that the induced force density must be of the form

$$\boldsymbol{F}_{\mathrm{ind}}(\boldsymbol{r}) = a^{-1}\boldsymbol{f}(\hat{\boldsymbol{r}})\delta(r-a)\,. \tag{2.10}$$

The factor a^{-1} is introduced here for convenience. If we use eq. (2.6) we can express the force $\boldsymbol{K}$, exerted by the fluid on the cylinder per unit of length, in terms of the induced force density One has

$$\boldsymbol{K} = -\int_{r=a} \mathrm{d}S\,\boldsymbol{P}(\boldsymbol{r})\cdot\hat{\boldsymbol{r}} = -\int_{r\leqslant a} \mathrm{d}\boldsymbol{r}\,\nabla\cdot\boldsymbol{P}(\boldsymbol{r}) = -\int \mathrm{d}\boldsymbol{r}\,\boldsymbol{F}_{\mathrm{ind}}(\boldsymbol{r})\,, \tag{2.11}$$

where use was also made of Gauss' theorem and of eq. (2.8).

In order to solve formally the equations of motion for the fluid we introduce two-dimensional Fourier transforms of e.g. the velocity field

$$v(k) = \int dr\, e^{-ik\cdot r} v(r)\,. \tag{2.12}$$

The equations of motion (2.6) and (2.7), combined with eq. (2.3) become in wavevector representation

$$i\rho U\cdot k v(k) + ikp(k) + \eta k^2 v(k) = F_{ind}(k)\,, \tag{2.13}$$

$$k\cdot v(k) = 0\,. \tag{2.14}$$

Applying the operator $\mathbb{1} - \hat{k}\hat{k}$, where $\mathbb{1}$ is the two-dimensional unit tensor and $\hat{k} \equiv k/k$, to both sides of eq. (2.13), and using the definitions

$$\alpha \equiv \rho U/\eta, \quad \hat{U} \equiv U/U\,, \tag{2.15}$$

we get the algebraic equation

$$\eta(k^2 + i\alpha\hat{U}\cdot k)v(k) = (\mathbb{1} - \hat{k}\hat{k})\cdot F_{ind}(k)\,. \tag{2.16}$$

The formal solution of this equation is given by

$$v(k) = (2\pi)^2 U\delta(k) + [\eta(k^2 + i\alpha U\cdot k)]^{-1}(\mathbb{1} - \hat{k}\hat{k})\cdot F_{ind}(k)\,. \tag{2.17}$$

This solution already contains the information that the unperturbed fluid, i.e. the fluid in absence of the cylinder, moves with velocity U.

3. Derivation of a simple expression for the mobility

In view of the symmetry of the present problem we can express the relation between the velocity U of the fluid at infinity and the force K exerted on the cylinder per unit of length as

$$U = \mu K\,. \tag{3.1}$$

In our subsequent analysis we shall derive an explicit expression for the (translational) mobility μ, which is a function of the Reynolds number R, defined as

$$R \equiv \rho U a/\eta = \alpha a\,. \tag{3.2}$$

To achieve this we expand in section 4 the induced force density in irreducible force multipoles. From the definition of these multipoles (cf. eqs. (4.5) and (4.6)) it follows that if one wants to retain only the zeroth force multipole, one may equivalently consider $f(\hat{r})$ in eq. (2.10) to be independent of $\hat{r}$. We shall first make this simplifying assumption and derive on this basis a simple expression for μ. Since there are only two relevant unit vectors in this problem, viz. $\hat{r}$ and $\hat{U}$, the induced force may in that case be written as

$$F_{ind}(r) = -\eta a^{-1} g(R) U \delta(r-a) \tag{3.3}$$

with $g(R)$ a dimensionless function of R. In wavevector representation this relation becomes

$$F_{ind}(k) = -2\pi\eta g(R) J_0(ka) U, \tag{3.4}$$

where

$$J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \, e^{\pm ix\cos\varphi} \tag{3.5}$$

is the Bessel function of the first kind of order zero with argument x. It follows from eqs. (2.11) and (3.3) that

$$K = 2\pi\eta g(R) U = \mu^{-1} U. \tag{3.6}$$

We shall now make use of the following identity for the "surface" average of the velocity field:

$$\begin{aligned} \overline{v(r)}^{S} &\equiv \frac{1}{2\pi a} \int dr \, v(r) \delta(r-a) \\ &= \frac{1}{4\pi^2} \int dk \, \overline{e^{ik\cdot r}}^{S} v(k) \\ &= \frac{1}{4\pi^2} \int dk \, J_0(ka) v(k). \end{aligned} \tag{3.7}$$

If we insert eq. (3.4) in eq. (2.17), and the resulting equation in the last member of eq. (3.7), while we also use eq. (2.5), we obtain:

$$\begin{aligned} U &= \frac{g(R)}{2\pi} \int dk [J_0(ka)]^2 (k^2 + i\alpha \hat{U}\cdot k)^{-1} (1 - \hat{k}\hat{k}) \cdot U \\ &= \frac{g(R)}{2\pi} \int d\hat{k} (1 - \hat{k}\hat{k}) \cdot U \int_0^\infty dk \, [J_0(ka)]^2 (k + i\alpha \hat{U}\cdot\hat{k})^{-1}. \end{aligned} \tag{3.8}$$

Since the imaginary part of the integrand vanishes upon integration over $\hat{k}$, this equation becomes

$$U = \frac{g(R)}{2\pi} \int d\hat{k} \, (1 - \hat{k}\hat{k}) \cdot U \int_0^\infty d(ka) \, ka [J_0(ka)]^2 [(ka)^2 + R^2 (\hat{U}\cdot\hat{k})^2]^{-1}. \tag{3.9}$$

The integration over ka may be performed (e.g. ref. 10). One then finds from eq.

(3.9) by taking on both sides the scalar product with $\hat{U}$, for the function $g(R)$ the result

$$(g(R))^{-1} = \frac{2}{\pi}\int_0^1 \mathrm{d}\xi\,\sqrt{1-\xi^2}\,I_0(R\xi)K_0(R\xi)\,. \tag{3.10}$$

Here we have introduced the variable $\xi \equiv \hat{U}\cdot\hat{k}$; $I_0(x)$ and $K_0(x)$ are the modified Bessel functions of the first and second kind, respectively, of order zero with argument x.

For the mobility we thus find

$$\mu = \pi^{-2}\eta^{-1}\int_0^1 \mathrm{d}\xi\,\sqrt{1-\xi^2}\,I_0(R\xi)K_0(R\xi)\,. \tag{3.11}$$

For low values of R, in particular for $R < 1$, we may expand $I_0(R\xi)$ and $K_0(R\xi)$ and retain only the lowest order terms

$$I_0(R\xi) = 1, \quad K_0(R\xi) = -\gamma - \ln\tfrac{1}{2}R\xi, \quad \gamma = 0.577\ldots, \tag{3.12}$$

γ is Euler's constant. Using these approximations one finds for the mobility

$$\mu = (4\pi\eta)^{-1}(\tfrac{1}{2} - \gamma - \ln\tfrac{1}{4}R) \tag{3.13}$$

which is the solution found by Lamb in 1911 [3]).

In the next section we shall extend the somewhat intuitive method we have used in this section, and develop a formal scheme for the calculation of μ. This scheme will enable one to calculate corrections to eq. (3.11), and will yield the complete solution as an infinite sum over products of integrals of the type (3.11). The comparison of the result (3.11) with results found by others can be found in section 5. It then turns out that eq. (3.11) already provides a fairly good description for the mobility over an extensive range of values of R.

4. Systematic evaluation of the mobility

In this section we shall derive a hierarchy of equations from which μ can in principle be calculated to any desired accuracy. To this end we evaluate the so-called velocity "surface" moments, defined as

$$\overline{\overleftrightarrow{\hat{r}^p}\boldsymbol{v}(\boldsymbol{r})}^{\,s} \equiv \frac{1}{2\pi a}\int \mathrm{d}\boldsymbol{r}\,\overleftrightarrow{\hat{r}^p}\boldsymbol{v}(\boldsymbol{r})\delta(r-a), \quad p\in\mathbb{N}\,. \tag{4.1}$$

Here, $\overleftrightarrow{\hat{r}^p}$ is the irreducible tensor of rank p, i.e. the tensor which is traceless and symmetric in all pairs of its indices, constructed with the vector $\hat{r}$. Such tensors are normalized by choosing the coefficient of $b_{\alpha_1} b_{\alpha_2} \ldots b_{\alpha_p}$ in the expression for $\overline{b_{\alpha_1} b_{\alpha_2} \ldots b_{\alpha_p}}$ equal to 1. For $p = 0, 1, 2, 3$ one has explicitly (note that the trace of $\mathbb{1}$ equals 2)

$$\overline{b^0} = 1, \quad \overline{b_\alpha} = b_\alpha, \quad \overline{b_\alpha b_\beta} = b_\alpha b_\beta - \tfrac{1}{2} b^2 \delta_{\alpha\beta},$$
$$\overline{b_\alpha b_\beta b_\gamma} = b_\alpha b_\beta b_\gamma - \tfrac{1}{4} b^2 (\delta_{\alpha\beta} b_\gamma + \delta_{\alpha\gamma} b_\beta + \delta_{\beta\gamma} b_\alpha) . \tag{4.2}$$

We show in appendix A that the following identity holds

$$\overline{\overline{\hat{r}^p} v(r)}^{S} = \frac{i^p}{4\pi^2} \int \mathrm{d}k \, \overline{\hat{k}^p} J_p(ka) v(k) , \tag{4.3}$$

where $J_p(x)$ is the Bessel function of the first kind of order p with argument x.

We now apply the boundary condition (2.5) to the left-hand side of eq. (4.3) and substitute eq. (2.17) in its right-hand side. We then obtain

$$-U\delta_{p0} = (2p)!! \, i^p (4\pi^2 \eta)^{-1} \int \mathrm{d}k \, (k^2 + i\alpha \hat{U} \cdot k)^{-1} J_p(ka) \overline{\hat{k}^p} (\mathbb{1} - \hat{k}\hat{k}) \cdot F_{\mathrm{ind}}(k) . \tag{4.4}$$

The factor $(2p)!! = 2^p p!$ has been introduced here for convenience.

It can be shown (see appendix A) that the induced force density $F_{\mathrm{ind}}(k)$ may be expanded as follows

$$F_{\mathrm{ind}}(k) = \sum_{l=0}^{\infty} (2l)!! (-i)^l J_l(ka) \overline{\hat{k}^l} \odot F^{(l+1)} , \tag{4.5}$$

where $F^{(l+1)}$ is the lth irreducible force multipole, defined as (cf. also eq. (2.10))

$$F^{(l+1)} \equiv \frac{1}{l!} \int \mathrm{d}\hat{r} \, \overline{\hat{r}^l} f(\hat{r})$$
$$= i^l a^{-l} (l!)^{-1} \left[\frac{\partial^l}{\partial k^l} F_{\mathrm{ind}}(k) \right]_{k=0} . \tag{4.6}$$

We note that this implies for the force:

$$K = -F^{(1)} . \tag{4.7}$$

In eq. (4.5), the symbol $\odot$ denotes the full l-fold contraction of the tensors $\overline{\hat{k}^l}$ and $F^{(l+1)}$, with the convention that the last index of $\overline{\hat{k}^l}$ is contracted with the first index of $F^{(l+1)}$, etc.

Upon substitution of eq. (4.5) in the right-hand side of eq. (4.4) we obtain

$$U\delta_{p0} = (4\pi\eta)^{-1} \sum_{l=0}^{\infty} (2\delta_{pl} - 1) B^{(p+1,l+1)} \odot F^{(l+1)} , \tag{4.8}$$

where the *connector* $B^{(p+1,l+1)}$ is given by

$$\boldsymbol{B}^{(p+1,l+1)} = (2p)!!(2l)!!(1-2\delta_{pl})\mathrm{i}^{p-l}\frac{1}{\pi}\int \mathrm{d}\hat{k}\,\overline{\hat{k}^{p}}(\mathbf{1}-\hat{k}\hat{k})\overline{\hat{k}^{l}}$$

$$\times \int_0^\infty \mathrm{d}k\, J_p(ka)J_l(ka)(k+\mathrm{i}\alpha\hat{U}\cdot\hat{k})^{-1}\,. \tag{4.9}$$

The factor $(1-2\delta_{pl})$ has been introduced for convenience.

The imaginary part of $\boldsymbol{B}^{(p+1,l+1)}$ vanishes upon integration over $\hat{k}$, and therefore $\boldsymbol{B}^{(p+1,l+1)}$ may be written as

$$\boldsymbol{B}^{(p+1,l+1)} = (2p)!!(2l)!!(1-2\delta_{pl})\frac{2}{\pi}\int_{\xi\geq 0} \mathrm{d}\hat{k}\,\overline{\hat{k}^{p}}(\mathbf{1}-\hat{k}\hat{k})\overline{\hat{k}^{l}}\,B^{(p,l)} \tag{4.10}$$

with $\xi \equiv \hat{U}\cdot\hat{k}$ (cf. eq. (3.10)). The (real) scalar function $B^{(p,l)}$ is given by

$$B^{(p,l)} = \mathrm{Re}\,\mathrm{i}^{p-l}\int_0^\infty \mathrm{d}k\, J_p(ka)J_l(ka)(k+\mathrm{i}\alpha\hat{U}\cdot\hat{k})^{-1}\,, \tag{4.11}$$

where Re denotes the real part. The integration over k may be carried out (cf. Gradshteyn and Ryzhik[10])) and yields

$$B^{(p,l)} = (-1)^{(l-p)\theta(l-p)} I_{\max(p,l)}(R\xi)K_{\min(p,l)}(R\xi)\,. \tag{4.12}$$

In eq. (4.12) $I_n(x)$ and $K_n(x)$ are the modified Bessel functions of the first and second kind, respectively, of order n with argument x; $\max(p,l)$ and $\min(p,l)$ denote the larger and smaller integer, respectively, of p and l; $\theta(l-p)$ is the Heaviside function, defined as

$$\theta(l-p) = \begin{cases} 0 & \text{if } l \leq p\,, \\ 1 & \text{if } l > p\,. \end{cases} \tag{4.13}$$

The connectors $\boldsymbol{B}^{(m,n)}$ have the following symmetry property:

$$\boldsymbol{B}^{(m,n)} = (-1)^{m+n}\tilde{\boldsymbol{B}}^{(n,m)}\,, \tag{4.14}$$

where $\tilde{\boldsymbol{B}}$ is the generalized transposed of the tensor $\boldsymbol{B}$ of rank $q = m+n$, defined by

$$(\tilde{\boldsymbol{B}})_{\alpha_1,\alpha_2,\ldots,\alpha_q} \equiv (\boldsymbol{B})_{\alpha_q,\ldots,\alpha_2,\alpha_1}\,. \tag{4.15}$$

For symmetry reasons the fluid cannot exert a torque, $\boldsymbol{T}$, on the cylinder. The torque can be expressed in terms of the induced force density as follows

$$T = -\int_{r=a} \mathrm{d}S\, \boldsymbol{r} \wedge \boldsymbol{P}(\boldsymbol{r})\cdot\hat{\boldsymbol{r}} = -\int_{r\leqslant a} \mathrm{d}\boldsymbol{r}\, \boldsymbol{r} \wedge (\boldsymbol{F}_{\mathrm{ind}}(\boldsymbol{r}) + \rho \boldsymbol{U}\cdot\nabla\boldsymbol{v}(\boldsymbol{r})) = a\epsilon : \boldsymbol{F}^{(2)}, \tag{4.16}$$

where ϵ is the Levi–Civita tensor, and where eqs. (2.6), (2.8), (2.10) and (4.6) have been used. It follows, since $\boldsymbol{T} = 0$, that the multipole $\boldsymbol{F}^{(2)}$ is symmetric. Consequently all connectors $\boldsymbol{B}^{(m,2)}$ in eq. (4.8) may be symmetrized with respect to their last two indices. From now on $\boldsymbol{B}^{(m,2)}$ will therefore denote its symmetric part.

With the help of eq. (4.7) we rewrite eq. (4.8) in the desired form of a hierarchy for the force multipoles:

$$4\pi\eta\boldsymbol{U} = -\boldsymbol{B}^{(1,1)}\cdot\boldsymbol{K} - \sum_{m=2}^{\infty} \boldsymbol{B}^{(1,m)} \odot \boldsymbol{F}^{(m)}, \tag{4.17}$$

$$\boldsymbol{F}^{(n)} = \boldsymbol{B}^{(n,n)^{-1}} \odot \left(-\boldsymbol{B}^{(n,1)}\cdot\boldsymbol{K} + \sum_{\substack{m=2\\ m\neq n}}^{\infty} \boldsymbol{B}^{(n,m)} \odot \boldsymbol{F}^{(m)} \right), \quad n \geqslant 2, \tag{4.18}$$

where $\boldsymbol{B}^{(n,n)^{-1}}$ is the generalized inverse of $\boldsymbol{B}^{(n,n)}$, only defined if acting on tensors of rank n which are irreducible in their first $n-1$ indices.

By iteration we can eliminate the higher force multipoles from the right-hand side of eq. (4.18) in favour of $\boldsymbol{K}$. When the resulting equations are substituted in eq. (4.17), we get an equation of the form (3.1), yielding for μ the expression

$$\mu = (4\pi\eta)^{-1}\hat{\boldsymbol{U}}\cdot\left[-\boldsymbol{B}^{(1,1)} + \sum_{s=1}^{\infty}\left(\sum_{m_1=2}^{\infty} \cdots \sum_{\substack{m_s=2\\ m_s\neq m_{s-1}}}^{\infty} \boldsymbol{B}^{(1,m_1)} \odot \boldsymbol{B}^{(m_1,m_1)^{-1}} \odot \boldsymbol{B}^{(m_1,m_2)} \odot \ldots \odot \boldsymbol{B}^{(m_s,1)} \right)\right]\cdot\hat{\boldsymbol{U}}. \tag{4.19}$$

We shall now discuss the contribution of each term in eq. (4.19) to the mobility for small values of the Reynolds number. From the standard developments of the Bessel functions (see appendix C) for small arguments it follows that the connectors behave as

$$\boldsymbol{B}^{(m,n)} = \begin{cases} \mathcal{O}(\ln R) & \text{for } m = n = 1, \\ \mathcal{O}(R^{|m-n|}) & \text{for } m \neq 1 \text{ and/or } n \neq 1. \end{cases} \tag{4.20}$$

Hence each term in eq. (4.19), which is a product of $s+1$ connectors, $s \geqslant 1$, gives a contribution to μ of order R^M, where M is an even power given by

$$M = m_1 + m_s - 2 + \sum_{j=2}^{s} |m_j - m_{j-1}|, \quad m_i \geqslant 2,\ i = 1, \ldots, s \tag{4.21}$$

while the first term ($s = 0$) is logarithmic in R. This term is dominant for very low values of R. For slightly higher values of R the term $\hat{U}\cdot \boldsymbol{B}^{(1,2)}:\boldsymbol{B}^{(2,2)^{-1}}:\boldsymbol{B}^{(2,1)}\cdot\hat{U}$ represents the most important correction to the first term. It is easily seen that

$$\mu = (4\pi\eta)^{-1}\hat{U}\cdot(-\boldsymbol{B}^{(1,1)} + \boldsymbol{B}^{(1,2)}:\boldsymbol{B}^{(2,2)^{-1}}:\boldsymbol{B}^{(2,1)})\cdot\hat{U} \tag{4.22}$$

is the expression for the mobility which is correct up to order R^3 and $R^3 \ln R$.

This expression takes into account the influence of the zeroth and first force multipoles and neglects all higher multipoles. We shall now assume that these higher multipoles may be neglected even for large values of R, and that eq. (4.22) remains a good approximation for μ.

We show in appendix B that the right-hand side of eq. (4.22) can be expressed in terms of three integrals:

$$\mu = \pi^{-2}\eta^{-1}\left[\int_0^1 \mathrm{d}\xi\,\sqrt{1-\xi^2}\,I_0(R\xi)K_0(R\xi) + \left[\int_0^1 \mathrm{d}\xi\,\xi\sqrt{1-\xi^2}\,I_1(R\xi)K_0(R\xi)\right]^2 \times \left[\int_0^1 \mathrm{d}\xi\,\xi^2\sqrt{1-\xi^2}\,I_1(R\xi)K_1(R\xi)\right]^{-1}\right]. \tag{4.23}$$

Numerical values for the so-called drag coefficient, calculated from this expression, can be found in table I in section 5.

We finally remark that if one wishes to include in the expression for μ the entire influence of e.g. the quadrupole moment, one should resum in the infinite series in eq. (4.19) all terms with $m_i \leqslant 3 (i = 1, 2, \ldots, s)$. Alternatively one might truncate the hierarchy (4.17) and (4.18) at $m = n = 3$ and solve μ from the remaining set of equations.

5. Discussion

It is customary, when discussing hydrodynamic forces exerted on an immersed cylinder (or spheroid) to introduce the drag coefficient C_D which is defined, in terms of the mobility, as

$$C_\mathrm{D} \equiv (\rho a \mu U)^{-1} = (\eta R \mu)^{-1}\,. \tag{5.1}$$

Let us now compare our results, eqs. (3.11) and (4.23) – which correspond to two successive orders of approximation – to results in closed form obtained previously by various authors. We shall also compare all these results to values for C_D obtained from numerical solution of the Oseen equation on the one hand, and to experimental values from measurements by Tritton[11]) on the other.

The following solutions for C_D in closed form have been found earlier:

H. Lamb (1911)

$$C_D = \frac{4\pi}{R}\left(\tfrac{1}{2} - \gamma - \ln \tfrac{1}{4}R\right)^{-1}, \tag{5.2}$$

L. Bairstow et al. (1923)

$$C_D = \frac{4\pi}{R}\left(I_0(\tfrac{1}{2}R)K_0(\tfrac{1}{2}R) + I_1(\tfrac{1}{2}R)K_1(\tfrac{1}{2}R)\right)^{-1}, \tag{5.3}$$

S. Kaplun (1957)[12]

$$C_D = \frac{4\pi}{R}\left[\left(\tfrac{1}{2} - \gamma - \ln \tfrac{1}{4}R\right)^{-1} - 0.87\left(\tfrac{1}{2} - \gamma - \ln \tfrac{1}{4}R\right)^{-3}\right]^{-1}. \tag{5.4}$$

To these results the following comments may be added. As is well known, Lamb's classic solution diverges for $R \approx 4$, and may only be expected to yield reasonable values for $R < 1$. Bairstow, in his derivation, has made use of a rather doubtful expansion for the velocity field and an equally doubtful way for solving for the coefficients in this expansion[13]. He obtains for $R \to \infty$ the limiting value $C_D = 2\pi$. Whereas Lamb's and Bairstow's solutions are based on the Oseen equation*, the expression of Kaplun was found by matching in an appropriate way the velocity field solution of the Stokes equation close to the cylinder to the solution of the Oseen equation. This procedure leads again to a divergence for C_D at $R \approx 4$, and yields for $R > 1.5$ physically irrelevant values for C_D.

From our equations (3.11) and (4.23) one finds for C_D

$$C_D = \frac{\pi^2}{R}\left[\int_0^1 d\xi\, \sqrt{1-\xi^2}\, I_0(R\xi)K_0(R\xi)\right]^{-1}, \tag{5.5}$$

$$C_D = \frac{\pi^2}{R}\left[\int_0^1 d\xi\, \sqrt{1-\xi^2}\, I_0(R\xi)K_0(R\xi) + \left[\int_0^1 d\xi\, \xi\sqrt{1-\xi^2}\, I_1(R\xi)K_0(R\xi)\right]^2 \right.$$
$$\left. \times \left[\int_0^1 d\xi\, \xi^2\sqrt{1-\xi^2}\, I_1(R\xi)K_1(R\xi)\right]^{-1}\right]^{-1}. \tag{5.6}$$

* Faxén[5] has given an exact solution for C_D in the form of the quotient of two infinite determinants. This form is not convenient for a practical evaluation of the drag coefficient. R. Berker has given a first approximation of Faxén's solution, which turns out to correspond also to an approximation of Bairstow's result. The Faxén–Berker expression diverges for $R \approx 7$ and improves Lamb's result only marginally. For this reason we have omitted the values obtained from this expression in table I.

In table I we have listed for values of R in the range $R = 0.2$ to $R = 100.0$ the corresponding values of C_D for the five expressions (5.2) to (5.6). Also listed in this table (in column (5.7)) are the four values for C_D obtained by Tomotika and Aoi[9]) from a "full numerical solution of the linearized Oseen equation"[14])*.

It is seen that Bairstow's result deviates substantially from Tomotika and Aoi's numerical values for $R \geqslant 2$. On the other hand, while our first approximation (5.5) gives already much better values than Bairstow's formula, our improved approximation, taking into account both the zeroth and first induced force multipole moments, fits the numerical results (5.7) remarkably well. This is also clear from fig. 1, in which C_D is plotted versus R for the various sets of values contained in table I. We have included for reference in this figure the curve which has been fitted to the measurements by Tritton[11])□.

A few remarks are in order. We have seen in section 4 that the expression for the mobility, calculated on the basis of the zeroth and first multipoles alone in the expansion in irreducible multipoles of the induced force density is correct up to and including order R^3 and $R^3 \ln R$ and will therefore certainly yield good values for C_D at low Reynolds numbers. One might wonder why the values for C_D remain

Table I

This table lists values of C_D for Lamb's (5.2), Bairstow's (5.3), Kaplun's (5.4), our first (5.5) and our second result (5.6), and from Tomotika and Aoi's solution (5.7)

R	(5.2)	(5.3)	(5.4)	(5.5)	(5.6)	(5.7)
0.2	21.53	21.47	19.33	21.43	21.25	
0.4	14.12	13.96	11.64	13.87	13.56	13.56
0.6	11.51	11.22	8.49	11.06	10.66	
0.8	10.25	9.79	6.45	9.56	9.09	
1.0	9.60	8.91	4.73	8.62	8.29	8.08
1.5	9.27	7.75	– 0.61	7.28	6.63	
2.0	10.20	7.20		6.55	5.84	5.84
3.0	19.90	6.71		5.75	4.96	
4.0	– 40.68	6.52		5.30	4.47	4.48
5.0		6.43		5.00	4.15	
10.0		6.31		4.24	3.42	
50.0		6.30		3.15	2.51	
100.0		6.29		2.84	2.28	

* In the last three decades numerous computer calculations on the basis of the full Navier–Stokes equation (and hence not directly relevant to our problem) have been published. For a survey of these articles we refer to the recent paper by Fornberg[15]).

□ Note that in the interval $0 < R \leqslant 0.4$ Kaplun's solution fits the experimental data much better than any solution of the Oseen equation.

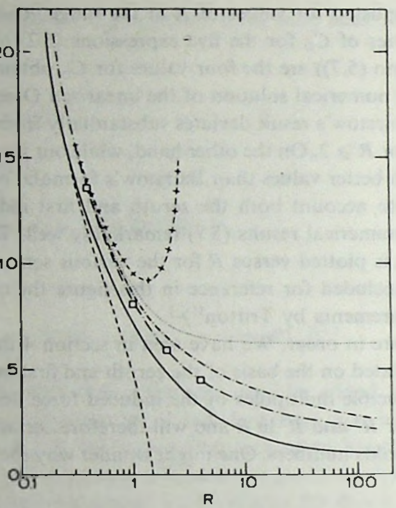

Fig. 1. C_D as function of R from experiment (——), for Lamb's (+ — + —), Bairstow's (··········), Kaplun's (----), our first (·—·—·) and our second (—·—) approximate solution. The results of Tomotika and Aoi are located in the centers of the small squares.

satisfactory for Reynolds numbers of order 10 and probably much higher. For this fact, which implies that higher multipoles do not contribute appreciably to the mobility even at rather high R, we have not succeeded to find a convincing explanation.

We note furthermore (see appendix C) that expressions (5.5) and (5.6) both tend to zero as $(\ln R)^{-1}$ for R tending to infinity. Whether this is the correct asymptotic behaviour of the Oseen drag on a cylinder is to some extent an open question†.

In a subsequent paper we shall apply the same methods to the analysis of the Oseen drag exerted on a sphere.

Appendix A. Irreducible multipole expansion and velocity surface moments

In this appendix we shall derive for the induced force density $F_{ind}(k)$ the expansion (4.5) and for the velocity surface moments the expression (4.3). We first

† Zeilon[16]) has solved the Oseen equation for the limit $\eta \downarrow 0 (R \to \infty)$, by assuming potential flow and by imposing different boundary conditions at the anterior and posterior side of the cylinder. He found $C_D = 1.314$.

prove the following identity

$$\overline{\frac{\partial^l}{\partial \boldsymbol{k}^l}} f(k) = \overline{\hat{\boldsymbol{k}}^l}\left(\frac{1}{k}\frac{\partial}{\partial k}\right)^l f(k), \quad l \in \mathbb{N}, \tag{A.1}$$

where $f(k)$ is an arbitrary function of k, the scalar part of $\boldsymbol{k}$. The identity

$$\frac{\partial}{\partial \boldsymbol{k}} = \boldsymbol{k}\frac{1}{k}\frac{\partial}{\partial k} + \frac{\mathbb{1} - \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}}{k} \cdot \frac{\partial}{\partial \hat{\boldsymbol{k}}} \tag{A.2}$$

proves (A.1) for the case $l = 1$. Since by definition

$$\overline{\frac{\partial^{l+1}}{\partial \boldsymbol{k}^{l+1}}} f(k) = \overline{\frac{\partial}{\partial \boldsymbol{k}}\overline{\frac{\partial^l}{\partial \boldsymbol{k}^l}}} f(k), \tag{A.3}$$

(A.1) follows for all l by virtue of complete induction.

One easily verifies (cf. Erdélyi et al.[17]) that the following relation holds

$$\overline{\hat{\boldsymbol{r}}^l} \odot \overline{\hat{\boldsymbol{r}}'^l} = \begin{cases} 1 & \text{for } l = 0, \\ 2^{1-l}\cos(l \arccos \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}') & \text{for } l \geqslant 1. \end{cases} \tag{A.4}$$

Furthermore

$$\delta(\hat{\boldsymbol{r}} - \hat{\boldsymbol{r}}') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im \arccos \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}'} \tag{A.5}$$

(cf. Jackson[18]). Combination of (A.4) and (A.5) yields the relation

$$\delta(\hat{\boldsymbol{r}} - \hat{\boldsymbol{r}}') = \frac{1}{2\pi} \sum_{l=0}^{\infty} 2^l \overline{\hat{\boldsymbol{r}}^l} \odot \overline{\hat{\boldsymbol{r}}'^l}. \tag{A.6}$$

Upon appropriate use of eqs. (2.10), (3.5), (4.6), (A.1) and (A.6), and the recursion relation

$$J_l(x) = (-x)^l \left(\frac{\mathrm{d}}{x\,\mathrm{d}x}\right)^l J_0(x), \tag{A.7}$$

one obtains

$$\begin{aligned} \boldsymbol{F}_{\text{ind}}(\boldsymbol{k}) &= \int \mathrm{d}\boldsymbol{r}\; e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \boldsymbol{F}_{\text{ind}}(\boldsymbol{r}) \\ &= \int \mathrm{d}\hat{\boldsymbol{r}}\; e^{-ia\boldsymbol{k}\cdot\hat{\boldsymbol{r}}} \boldsymbol{f}(\hat{\boldsymbol{r}}) \\ &= \int \mathrm{d}\hat{\boldsymbol{r}} \int \mathrm{d}\hat{\boldsymbol{r}}'\; \delta(\hat{\boldsymbol{r}} - \hat{\boldsymbol{r}}')\; e^{-ia\boldsymbol{k}\cdot\hat{\boldsymbol{r}}'} \boldsymbol{f}(\hat{\boldsymbol{r}}) \end{aligned}$$

$$= \frac{1}{2\pi}\sum_{l=0}^{\infty} 2^l \int d\hat{r}'\, e^{-iak\cdot\hat{r}'}\overline{\hat{r}'^l} \odot \int d\hat{r}\,\overline{\hat{r}^l} f(\hat{r})$$

$$= \sum_{l=0}^{\infty} 2^l l! \left(\frac{i}{a}\right)^l \left[\frac{\partial^l}{\partial k^l}\frac{1}{2\pi}\int d\hat{r}'\, e^{-iak\cdot\hat{r}'}\right] \odot F^{(l+1)}$$

$$= \sum_{l=0}^{\infty} (2l)!!\, i^l (ka)^l \left(\frac{1}{ka}\frac{\partial}{\partial ka}\right)^l J_0(ka)\overline{\hat{k}^l} \odot F^{(l+1)}$$

$$= \sum_{l=0}^{\infty} (2l)!!(-i)^l J_l(ka)\overline{\hat{k}^l} \odot F^{(l+1)} . \qquad \text{(A.8)}$$

Along similar lines one obtains upon combination of eqs. (3.5), (4.1), (A.1) and (A.7) for the velocity surface moments the expression

$$\overline{\overline{\hat{r}^p} v(r)}^{S} = \frac{1}{2\pi a}\int dr\, \overline{\hat{r}^p}\,\delta(r-a)\frac{1}{4\pi^2}\int dk\, e^{ik\cdot r} v(k)$$

$$= \frac{1}{4\pi^2}\int dk\, \overline{\overline{\hat{r}^p} e^{ik\cdot r}}^{S}\, v(k)$$

$$= \frac{1}{4\pi^2}\int dk \left(-\frac{i}{a}\right)^p \frac{\overline{\partial^p}}{\partial k^p} J_0(ka) v(k)$$

$$= \frac{i^p}{4\pi^2}\int dk\, J_p(ka)\overline{\hat{k}^p}\, v(k) . \qquad \text{(A.9)}$$

Appendix B. Proof of eq. (4.23)

We show in this appendix that the evaluation of eq. (4.22) in terms of integrals leads to the expression for μ given in eq. (4.23). We can write $B^{(1,1)}$ in the form

$$B^{(1,1)} = b_1\hat{U}\hat{U} + b_2(1 - \hat{U}\hat{U}) . \qquad \text{(B.1)}$$

Combined with eqs. (4.10) and (4.12) this yields for the first term at the right-hand side of eq. (4.22) (cf. also section 3)

$$-\hat{U}\cdot B^{(1,1)}\cdot\hat{U} = -b_1 = \frac{4}{\pi}\int_0^1 d\xi\, \sqrt{1-\xi^2}\, I_0(R\xi)K_0(R\xi) . \qquad \text{(B.2)}$$

Since $B^{(1,2)}$ is traceless and, according to our convention in section 4, symmetric in its last two indices, this connector must be of the form

$$B^{(1,2)} = b_3\hat{U}\overline{\hat{U}\hat{U}} + b_4\tfrac{1}{2}(1\hat{U} + \underset{\smile}{.\hat{U}.} - \hat{U}1) \qquad \text{(B.3)}$$

with $(\underset{\smile}{.\hat{U}.})_{\alpha\beta\gamma} \equiv \hat{U}_\beta\delta_{\alpha\gamma}$ and

$$b_3 + b_4 = 2\hat{U}\cdot \boldsymbol{B}^{(1,2)}:\overline{\hat{U}\hat{U}} = -\frac{16}{\pi}\int_0^1 \mathrm{d}\xi\, \xi\sqrt{1-\xi^2}\, I_1(R\xi)K_0(R\xi)\,. \tag{B.4}$$

One verifies immediately that $\boldsymbol{B}^{(2a,1)}\cdot\hat{U} = 0$, where 2a denotes the part of $\boldsymbol{B}^{(2,1)}$ which is antisymmetric in its first two indices ($\boldsymbol{B}^{(2a,1)}$ must be of the form $\hat{U}\mathbb{1} - \underset{\smile}{\cdot\hat{U}\cdot}$). For $\boldsymbol{B}^{(2s,1)}$, the symmetric part of $\boldsymbol{B}^{(2,1)}$ with respect to the first two indices, one finds from the symmetry property (4.14) and eq. (B.3)

$$\boldsymbol{B}^{(2s,1)} = -b_3\overline{\hat{U}\hat{U}}\hat{U} - b_4\tfrac{1}{2}(\hat{U}\mathbb{1} + \underset{\smile}{\cdot\hat{U}\cdot} - \mathbb{1}\hat{U})\,. \tag{B.5}$$

Next we consider $\boldsymbol{B}^{(2s,2s)}$, which must be of the form (cf. Mazur and Van Saarloos[8]), appendix E)

$$\boldsymbol{B}^{(2s,2s)} = 2b_5\boldsymbol{\Delta} + 4b_6\,\overline{\hat{U}\hat{U}}\,\overline{\hat{U}\hat{U}} + b_7\boldsymbol{D}(\hat{U}) \tag{B.6}$$

with

$$b_5 + b_6 = \overline{\hat{U}\hat{U}}:\boldsymbol{B}^{(2s,2s)}:\overline{\hat{U}\hat{U}}$$

$$= -\frac{16}{\pi}\int_0^1 \mathrm{d}\xi\, \xi^2\sqrt{1-\xi^2}\, I_1(R\xi)K_1(R\xi)\,. \tag{B.7}$$

In eq. (B.6) $\boldsymbol{\Delta}$ and $\boldsymbol{D}(\hat{U})$ denote the standard tensors of rank four:

$$\boldsymbol{\Delta} = \tfrac{1}{2}(\underset{\smile}{\cdot\cdot\cdot\cdot} + \underset{\smile}{\cdot\cdot\cdot\cdot} - \mathbb{1})\,, \tag{B.8}$$

$$\boldsymbol{D}(\hat{U}) = 2\hat{U}\hat{U}\hat{U}\hat{U} - \tfrac{1}{2}(\hat{U}\mathbb{1}\hat{U} + \underset{\smile}{\cdot\hat{U}\cdot}\,\hat{U} + \hat{U}\,\underset{\smile}{\cdot\hat{U}\cdot} + \underset{\smile}{\cdot\hat{U}\hat{U}\cdot})\,. \tag{B.9}$$

From the identity

$$\overline{\hat{U}\hat{U}} = \boldsymbol{B}^{(2s,2s)^{-1}}:\boldsymbol{B}^{(2s,2s)}:\overline{\hat{U}\hat{U}} \tag{B.10}$$

combined with the fact that

$$\boldsymbol{B}^{(2s,2s)}:\overline{\hat{U}\hat{U}} = 2(b_5 + b_6)\overline{\hat{U}\hat{U}} \tag{B.11}$$

it follows in a straightforward way that

$$\hat{U}\cdot\boldsymbol{B}^{(1,2)}:\boldsymbol{B}^{(2,2)^{-1}}:\boldsymbol{B}^{(2,1)}\cdot\hat{U} = \hat{U}\cdot\boldsymbol{B}^{(1,2s)}:\boldsymbol{B}^{(2s,2s)^{-1}}:\boldsymbol{B}^{(2s,1)}\cdot\hat{U}$$

$$= -(b_3+b_4)^2\,\overline{\hat{U}\hat{U}}:\boldsymbol{B}^{(2s,2s)^{-1}}:\overline{\hat{U}\hat{U}}$$

$$= -\tfrac{1}{4}(b_3+b_4)^2(b_5+b_6)^{-1}$$

$$= \frac{4}{\pi}\left[\int_0^1 \mathrm{d}\xi\,\xi\sqrt{1-\xi^2}\,I_1(R\xi)K_0(R\xi)\right]^2\left[\int_0^1 \mathrm{d}\xi\,\xi^2\sqrt{1-\xi^2}\,I_1(R\xi)K_1(R\xi)\right]^{-1} \tag{B.12}$$

which, together with eq. (B.2), proves the validity of eq. (4.23).

Appendix C. Asymptotic behaviour of the connectors

For the modified Bessel functions of the first and second kind, $I_n(x)$ and $K_n(x)$, one has the developments

$$I_n(x) = \sum_{m=0}^{\infty} A_{mn} x^{2m+n}, \tag{C.1}$$

$$K_n(x) = \sum_{m=0}^{n-1} B_{mn} x^{2m-n} + \sum_{p=0}^{\infty} C_{pn} x^{2p+n} (D_{pn} + \ln x), \tag{C.2}$$

where $B_{m0} = D_{p0} = 0$. With eqs. (4.10) and (4.12) one immediately shows that the behaviour for $R < 1$ of the connectors $\boldsymbol{B}^{(m,n)}$ is the one given in eq. (4.20).

We shall now discuss the asymptotic behaviour for large values of R of the integrals appearing in eq. (5.6) (which are related to the connectors $\boldsymbol{B}^{(1,1)}$, $\boldsymbol{B}^{(1,2)}$ and $\boldsymbol{B}^{(2,2)}$). For large values of x one has for all $m, n \geqslant 0$:

$$I_m(x) K_n(x) = \frac{1}{2x} + \mathcal{O}(x^{-3}). \tag{C.3}$$

It follows that for $R \gg 1$

$$\int_0^1 \mathrm{d}\xi\, \xi \sqrt{1-\xi^2}\, I_1(R\xi) K_0(R\xi) = \frac{\pi}{8R} + \mathcal{O}(R^{-3}), \tag{C.4}$$

$$\int_0^1 \mathrm{d}\xi\, \xi^2 \sqrt{1-\xi^2}\, I_1(R\xi) K_1(R\xi) = \frac{1}{6R} + \mathcal{O}(R^{-3}). \tag{C.5}$$

For the third integral in eq. (5.6) one has

$$R \int_0^1 \mathrm{d}\xi \sqrt{1-\xi^2}\, I_0(R\xi) K_0(R\xi) = \left(\int_0^{\sqrt{R}} + \int_{\sqrt{R}}^{R} \right) \mathrm{d}x \sqrt{1-\left(\frac{x}{R}\right)^2}\, I_0(x) K_0(x). \tag{C.6}$$

The first integral at the right-hand side of eq. (C.6) is positive for all R. Since the integrand is sufficiently well-behaved at zero, this integral diverges at most as $\ln R$, due to the upper boundary. The second integral may be evaluated with eq. (C.3) and yields

$$\int_{\sqrt{R}}^{R} \mathrm{d}x \sqrt{1-\left(\frac{x}{R}\right)^2}\, I_0(x) K_0(x)$$

$$= \int_{\sqrt{R}}^{R} \mathrm{d}x \frac{1}{2x} \sqrt{1-\left(\frac{x}{R}\right)^2} + \int_{\sqrt{R}}^{R} \mathrm{d}x\, \mathcal{O}(x^{-3}) \sqrt{1-\left(\frac{x}{R}\right)^2} = N \ln R + M(R) \tag{C.7}$$

with

$$N = \mathcal{O}(1), \quad \lim_{R\to\infty} M(R) < \infty . \tag{C.8}$$

The statement at the end of section 5 that C_D as given by eqs. (5.5) and (5.6) tends to zero as $(\ln R)^{-1}$ for $R\to\infty$ then follows.

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Chapter II

The Oseen drag on a sphere and the method of induced forces

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THE OSEEN DRAG ON A SPHERE AND THE METHOD OF INDUCED FORCES

1. Introduction

The best known expression for the relation between the drag force $\boldsymbol{K}$, exerted by a viscous fluid on a sphere with radius a and velocity $\boldsymbol{U}$ is Stokes' celebrated formula[1])

$$\boldsymbol{K} = -6\pi\eta a \boldsymbol{U} . \tag{1.1}$$

This formula was derived on the basis of the Stokes equation. In this equation all momentum convection has been neglected. In 1911 Oseen[2]) pointed out that this neglect could not be justified. To remove the objection he proposed a different linear equation. This substituting equation, today known as the Oseen equation, will be discussed in section 2.

The first approximate solution for the drag force, based on the Oseen equation, was found by Oseen[3]) himself in 1913; the construction of the exact solution is due to Goldstein[4]). Goldstein's solution may be cast in the form of a power series in the Reynolds number R. The convergence of this series is rather slow for $R > 1$. In 1970 Van Dyke[5]) has analysed the behaviour of the series after calculating numerically the coefficients of the first 24 terms.

In this paper we shall evaluate the Oseen drag exerted on a sphere by an alternative method. Our analysis of this problem is different from those mentioned above in sofar that it does not require explicit knowledge of the velocity and pressure field. The method we apply to this end is the same as the one we have used previously to analyse the Oseen drag on a circular cylinder[6]). This method is based on the introduction of an induced force density and its expansion in irreducible force multipoles, and is in fact an adapted form of the method used by Mazur and Van Saarloos[7]) to analyse many-sphere hydrodynamic interactions in Stokes flow. In section 3 we summarize the equations describing incompressible fluid flow past a sphere and we give the formal solution for the velocity field in wavevector representation. In section 4 we derive a hierarchy of equations for the irreducible force multipoles. By appropriate use of the boundary conditions and elimination of all higher force multipoles in favour of the force (the zeroth multipole), an exact expression is obtained for the drag force as function of the Reynolds number. In section 5 we compare the approximation to this exact expression which is obtained if only the zeroth and first force multipole are taken into account, to results proposed previously by various authors, as well as to Van Dyke's numerical results. The comparison indicates that this approximation yields values for the drag coefficient which, up to Reynolds numbers of order 10^2, differ by at most 10% from those obtained by numerical evaluation of Goldstein's solution (and by no more than 1% up to $R \approx 10$).

2. The Oseen equation

To describe incompressible stationary flow past e.g. a sphere we may use the Oseen equation

$$\rho \boldsymbol{U} \cdot \nabla \boldsymbol{v}(\boldsymbol{r}) + \nabla \cdot \boldsymbol{P}(\boldsymbol{r}) = 0 \qquad \text{for } r > a \tag{2.1}$$

$$\nabla \cdot \boldsymbol{v}(\boldsymbol{r}) = 0 \qquad \text{for } r > a \tag{2.2}$$

with

$$P_{\alpha\beta} = p\delta_{\alpha\beta} - \eta\left(\frac{\partial v_\alpha}{\partial r_\beta} + \frac{\partial v_\beta}{\partial r_\alpha}\right). \tag{2.3}$$

Here $\boldsymbol{v}(\boldsymbol{r})$ is the velocity field, $\boldsymbol{P}(\boldsymbol{r})$ the pressure tensor, $p(\boldsymbol{r})$ the hydrostatic pressure, $\boldsymbol{U}$ the fluid velocity at infinity and ρ and η the density and viscosity of the fluid, respectively.

The use of the Oseen equation rather than the Stokes equation may be understood by observing that at certain, for low Reynolds numbers large, distances from the sphere there are areas where $|\rho \boldsymbol{U} \cdot \nabla \boldsymbol{v}(\boldsymbol{r})|$ (which is essentially the convection term of the Navier–Stokes equation at large distance) is not smaller than $|\eta \Delta \boldsymbol{v}(\boldsymbol{r})|$. This remark is due to Oseen himself; he used for $\boldsymbol{v}(\boldsymbol{r})$ the solution of the Stokes equation. In 1957 Proudman and Pearson[8]) pointed out that the requirement of momentum conservation leads to the same conclusion. They noted that the perturbation of the incoming velocity field cannot in *all* directions diminish more rapidly than r^{-2} but that $|\rho \boldsymbol{U} \cdot \nabla \boldsymbol{v}(\boldsymbol{r})|$ remains smaller than $|\eta \Delta \boldsymbol{v}(\boldsymbol{r})|$

only if the decay is at least exponential. The distance at which $\rho \boldsymbol{U} \cdot \nabla \boldsymbol{v}(\boldsymbol{r})$ may no longer be neglected tends to infinity in the limit of zero Reynolds number $R \equiv \rho U a/\eta$. In this limit the drag force calculated with the Stokes equation is equal to the drag force which would follow from the full Navier–Stokes equation. But since zero Reynolds number corresponds to zero velocity, an expansion of the drag force in powers of R is necessary. According to Proudman and Pearson logarithms may also appear in this expansion. In 1938 Goldstein[9]) stated that the drag force calculated on the basis of the Oseen equation is up to first order in R identical with the solution based on the Navier–Stokes equation*. For fluids obeying this last equation the drag force should therefore for $R \ll 1$ be described by Oseen's solution

$$\boldsymbol{K} = -6\pi\eta a \boldsymbol{U}(1 + \tfrac{3}{8}R)\,. \tag{2.4}$$

The experiments of Maxworthy[10]) seem to confirm this fact.

One may arrive at the Oseen equation via a quite different line of thought. This way to derive the Oseen equation was perhaps hinted at by Oseen himself. If we consider a sphere moving with constant velocity $-\boldsymbol{U}$ through a fluid at rest at infinity, in a reference frame which is also at rest, the governing *linearized* equation is

$$\rho \frac{\partial \boldsymbol{v}(\boldsymbol{r}, t)}{\partial t} = -\nabla p(\boldsymbol{r}, t) + \eta \Delta \boldsymbol{v}(\boldsymbol{r}, t) \tag{2.5}$$

$$\text{for } |\boldsymbol{r} - \boldsymbol{R}(t)| > a\,,$$

$$\nabla \cdot \boldsymbol{v}(\boldsymbol{r}, t) = 0 \tag{2.6}$$

where $\boldsymbol{R}(t)$ denotes the position of the center of the sphere

$$\boldsymbol{R}(t) = \boldsymbol{R}(0) - \boldsymbol{U}t\,. \tag{2.7}$$

The coordinate transformation

$$t' = t, \quad \boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{U}t \tag{2.8}$$

implies that

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} + \boldsymbol{U} \cdot \nabla', \quad \nabla = \nabla'\,. \tag{2.9}$$

Upon application of this transformation to eqs. (2.5) and (2.6), and transformation of the velocity and pressure field according to

$$\boldsymbol{v}'(\boldsymbol{r}', t') = \boldsymbol{v}(\boldsymbol{r}', t') + \boldsymbol{U}, \quad p'(\boldsymbol{r}', t') = p(\boldsymbol{r}', t')\,, \tag{2.10}$$

* A proof of this statement was later given by Chester[11]).

we obtain the equation which describes the fluid motion past a sphere at rest. This situation is time independent, and the resulting equation is the Oseen equation, eq. (2.1). From the above considerations it will be clear that solving the linearized equation for a moving sphere, without neglecting the time dependence of the boundary condition, is equivalent to solving the Oseen equation for a sphere at rest.

3. Formulation of the problem

We consider a sphere with radius a, at rest in a viscous, incompressible, unbounded fluid. The fluid motion is assumed to obey the Oseen equation, eqs. (2.1)–(2.3). We apply stick boundary conditions at the surface of the sphere:

$$\boldsymbol{v}(\boldsymbol{r}) = 0 \quad \text{for } r = a\,. \tag{3.1}$$

The method of induced forces enables one to replace the set of equations (2.1)–(2.3) and (3.1) by an equivalent one in which the fluid equations are extended within the sphere and written in the form

$$\left.\begin{aligned} &\rho \boldsymbol{U}\cdot\nabla\boldsymbol{v}(\boldsymbol{r}) + \nabla\cdot\mathsf{P}(\boldsymbol{r}) = \boldsymbol{F}_{\mathrm{ind}}(\boldsymbol{r}) \qquad (3.2)\\ &\nabla\cdot\boldsymbol{v}(\boldsymbol{r}) = 0 \qquad (3.3)\end{aligned}\right|\ \text{for all } \boldsymbol{r}$$

with $\boldsymbol{F}_{\mathrm{ind}}(\boldsymbol{r}) = 0$ for $r > a$. The extension of the velocity field inside the sphere is chosen as

$$\boldsymbol{v}(\boldsymbol{r}) = 0 \quad \text{for } r \leqslant a\,. \tag{3.4}$$

On the hydrostatic pressure we impose the condition

$$p(\boldsymbol{r}) = 0 \quad \text{for } r < a\,. \tag{3.5}$$

It follows from substitution of eqs. (3.4) and (3.5) in eq. (3.2) that the induced force density must be of the form:

$$\boldsymbol{F}_{\mathrm{ind}}(\boldsymbol{r}) = a^{-2}\boldsymbol{f}(\hat{\boldsymbol{r}})\delta(r - a)\,. \tag{3.6}$$

Here $\hat{\boldsymbol{r}} \equiv \boldsymbol{r}/r$ denotes the unit vector normal to the surface and pointing in the outward direction. The factor a^{-2} has been introduced for convenience. With the aid of eqs. (3.2) and (3.4) we can express the force $\boldsymbol{K}$, exerted by the fluid on the sphere, in terms of the induced force density as follows

$$\boldsymbol{K} = -\int_{r=a} \mathrm{d}S\,\hat{\boldsymbol{r}}\cdot\mathsf{P}(\boldsymbol{r}) = -\int_{r\leqslant a} \mathrm{d}\boldsymbol{r}\,\nabla\cdot\mathsf{P}(\boldsymbol{r}) = -\int \mathrm{d}\boldsymbol{r}\,\boldsymbol{F}_{\mathrm{ind}}(\boldsymbol{r})\,. \tag{3.7}$$

We now define the Fourier transform of e.g. the velocity field by

$$v(k) \equiv \int dr\, e^{-ik\cdot r} v(r) . \tag{3.8}$$

The equations of motion, eqs. (3.2) and (3.3), combined with eq. (2.3), then become in wavevector representation

$$i\rho U \cdot k v(k) + ikp(k) + \eta k^2 v(k) = F_{ind}(k) , \tag{3.9}$$

$$k \cdot v(k) = 0 . \tag{3.10}$$

If one applies the operator $\mathbb{1} - \hat{k}\hat{k}$ to both sides of eq. (3.9) one obtains with eq. (3.10)

$$(\eta k^2 + i\rho U \cdot k) v(k) = (\mathbb{1} - \hat{k}\hat{k}) \cdot F_{ind}(k) , \tag{3.11}$$

where $\mathbb{1}$ is the unit tensor and $\hat{k} \equiv k/k$. If we use the definitions

$$\alpha \equiv \rho U/\eta, \quad \hat{U} \equiv U/U , \tag{3.12}$$

the formal solution of eq. (3.11) can be written as

$$v(k) = (2\pi)^3 U \delta(k) + [\eta(k^2 + i\alpha \hat{U} \cdot k)]^{-1} (\mathbb{1} - \hat{k}\hat{k}) \cdot F_{ind}(k) . \tag{3.13}$$

The term $(2\pi)^3 U \delta(k)$ is the solution for the velocity field of the unperturbed fluid (i.e. the fluid in absence of the sphere, $F_{ind}(k) = 0$).

4. Systematic evaluation of the mobility

The symmetry of the present problem allows us to express the relation between the velocity U of the fluid at infinity and the force K exerted on the sphere as

$$U = \mu K . \tag{4.1}$$

In this section a scheme will be developed for the systematic evaluation of the (translational) mobility μ as function of the Reynolds number. This scheme runs parallel to the scheme presented in our previous paper, on the Oseen friction of a circular cylinder.

As first step we introduce the so-called velocity surface moments, defined as

$$\overline{\overline{\hat{r}^p} v(r)}^{\,s} \equiv \frac{1}{4\pi a^2} \int dr\, \overline{\hat{r}^p} v(r) \delta(r - a), \quad p \in \mathbb{N} . \tag{4.2}$$

Here $\overline{\hat{r}^p}$ is the irreducible tensor of rank p, i.e. the tensor traceless and symmetric in any pair of its indices, constructed with the vector $\hat{r}$. For normalisation the coefficient of the term $b_{\alpha_1} \ldots b_{\alpha_p}$ in the expression for $\overline{b_{\alpha_1} \ldots b_{\alpha_p}}$ has been chosen

1 (see e.g. ref. 12). In appendix A we show that the following identity holds

$$\overline{\overline{\hat{r}^p v(r)}}^{S} = \frac{i^p}{(2\pi)^3} \int dk \left(\frac{\pi}{2ka}\right)^{1/2} J_{p+1/2}(ka)\overline{\hat{k}^p} v(k)\,, \tag{4.3}$$

where $J_n(x)$ is the Bessel function of the first kind of order n with argument x.

In appendix A we also show that the induced force density $F_{ind}(k)$ has the following expansion

$$F_{ind}(k) = \sum_{l=0}^{\infty} (2l+1)!!(-i)^l \left(\frac{\pi}{2ka}\right)^{1/2} J_{l+1/2}(ka)\overline{\hat{k}^l} \odot F^{(l+1)}\,, \tag{4.4}$$

$F^{(l+1)}$ is the lth irreducible force multipole, defined as

$$F^{(l+1)} \equiv (l!)^{-1} \int d\hat{r}\, \overline{\hat{r}^l} f(\hat{r}) = i^l a^{-l} (l!)^{-1} \left[\frac{\partial^l}{\partial k^l} F_{ind}(k)\right]_{k=0}\,. \tag{4.5}$$

The symbol $\odot$ denotes the full contraction of the tensors $\overline{\hat{k}^l}$ and $F^{(l+1)}$, with the convention that the last index of $\overline{\hat{k}^l}$ is contracted with the first index of $F^{(l+1)}$, etc; $(2l+1)!! \equiv (2l+1)(2l-1)\ldots 3 \times 1$. We note that for the case of the circular cylinder, i.e. in two dimensions, one has, instead of eqs. (4.3) and (4.4), analogous expressions containing, however, at their right-hand side Bessel functions of integer order instead of spherical ones. In view of eqs. (3.6), (3.7) and (4.5) we have

$$K = -F^{(1)}\,. \tag{4.6}$$

We now evaluate the left-hand side of eq. (4.3), using eq. (3.1); in its right-hand side we substitute eqs. (3.13) and (4.4). We then obtain the set of equations

$$-U\delta_{p0} = (4\pi\eta a)^{-1} \sum_{l=0}^{\infty} (1-2\delta_{pl}) B^{(p+1,l+1)} \odot F^{(l+1)} \tag{4.7}$$

with

$$B^{(p+1,l+1)} = (2p+1)!!(2l+1)!!(1-2\delta_{pl})\, i^{p-l} \frac{1}{4\pi} \int d\hat{k}\, \overline{\hat{k}^p} (\mathbb{1} - \hat{k}\hat{k}) \overline{\hat{k}^l}$$

$$\times \int_0^{\infty} dk\, J_{p+1/2}(ka) J_{l+1/2}(ka) (k + i\alpha \hat{U} \cdot \hat{k})^{-1}\,. \tag{4.8}$$

The factors $(2p+1)!!$ and $(1-2\delta_{pl})$ have been introduced for convenience.

The imaginary part of the connectors $B^{(p+1,l+1)}$ vanishes upon integration over $\hat{k}$, and therefore $B^{(p+1,l+1)}$ may be written as

$$B^{(p+1,l+1)} = (2p+1)!!(2l+1)!!(1-2\delta_{pl}) \frac{1}{4\pi} \int d\hat{k}\, \overline{\hat{k}^p} (\mathbb{1} - \hat{k}\hat{k}) \overline{\hat{k}^l} B^{(p,l)} \tag{4.9}$$

with $B^{(p,l)}$ the real scalar function given by (see e.g. ref. 13, 6.577)

$$B^{(p,l)} = \mathrm{Re}\, i^{p-l} \int_0^{\infty} \mathrm{d}k\, J_{p+1/2}(ka) J_{l+1/2}(ka)(k + \mathrm{i}\alpha \hat{U} \cdot \hat{k})^{-1}$$

$$= (-1)^{(l+p)\theta(l-p)} I_{(\max(p,l)+1/2)}(R\xi) K_{(\min(p,l)+1/2)}(R\xi) \,. \tag{4.10}$$

Re denotes the real part; $R = \alpha a = \rho U a/\eta$ is the Reynolds number and $\xi \equiv \hat{U} \cdot \hat{k}$. $I_n(x)$ and $K_n(x)$ are the modified Bessel functions of the first and second kind, respectively of order n with argument x; $\max(p, l)$ and $\min(p, l)$ denote the larger and smaller integer, respectively, of p and l. $\theta(l-p)$ is the Heaviside function, defined by

$$\theta(l-p) = \begin{cases} 0 & \text{if } l \leqslant p\,, \\ 1 & \text{if } l > p\,. \end{cases} \tag{4.11}$$

It is clear from eqs. (4.9) and (4.10) that the connectors satisfy the symmetry relation

$$\boldsymbol{B}^{(m,n)} = (-1)^{m+n} \tilde{\boldsymbol{B}}^{(n,m)} \,. \tag{4.12}$$

Here $\tilde{\boldsymbol{B}}$ is the generalized transposed of the tensor $\boldsymbol{B}$ of rank $q = m + n$, defined by

$$(\tilde{\boldsymbol{B}})_{\alpha_1,\ldots,\alpha_q} \equiv (\boldsymbol{B})_{\alpha_q,\ldots,\alpha_1} \,. \tag{4.13}$$

The torque $\boldsymbol{T}$, exerted by the fluid on the sphere, can with the aid of eqs. (3.2) and (3.4) be expressed in terms of the induced force density as

$$\boldsymbol{T} = -\int_{r=a} \mathrm{d}S\, \boldsymbol{r} \wedge \boldsymbol{P}(\boldsymbol{r}) \cdot \hat{\boldsymbol{r}}$$

$$= -\int \mathrm{d}\boldsymbol{r}\, \boldsymbol{r} \wedge (\boldsymbol{F}_{\mathrm{ind}}(\boldsymbol{r}) + \rho \boldsymbol{U} \cdot \nabla \boldsymbol{v}(\boldsymbol{r})) = a\epsilon : \boldsymbol{F}^{(2)} \,, \tag{4.14}$$

where ϵ is the Levi–Civita tensor. Since the torque is zero in view of the symmetry of the problem, $\boldsymbol{F}^{(2)}$ is a symmetric tensor. We may therefore consider all connectors $\boldsymbol{B}^{(m,2)}$ in eq. (4.7) to have been symmetrized in their last two indices.

We now rewrite eq. (4.7) in the desired form of a hierarchy of equations for the force multipoles with the help of eq. (4.6):

$$4\pi\eta a \boldsymbol{U} = -\boldsymbol{B}^{(1,1)} \cdot \boldsymbol{K} - \sum_{m=2}^{\infty} \boldsymbol{B}^{(1,m)} \odot \boldsymbol{F}^{(m)} \,, \tag{4.15}$$

$$F^{(n)} = B^{(n,n)^{-1}} \odot \left(-B^{(n,1)} \cdot K + \sum_{\substack{m=2 \\ m \neq n}}^{\infty} B^{(n,m)} \odot F^{(m)} \right) \quad n \geq 2, \tag{4.16}$$

where $B^{(n,n)^{-1}}$ is the generalized inverse of $B^{(n,n)}$, which is only defined if acting on tensors of rank n which are irreducible in their first $n-1$ indices.

By iteration we can eliminate the higher force multipoles from the right-hand side of eq. (4.16) in favour of K. When the resulting equations are substituted in eq. (4.15), we get an equation of the form (4.1), which determines μ:

$$\mu = (4\pi\eta a)^{-1} \hat{U} \cdot \Bigg[-B^{(1,1)} + \sum_{s=1}^{\infty} \Bigg(\sum_{m_1=2}^{\infty} \cdots \sum_{\substack{m_s=2 \\ m_s \neq m_{s-1}}}^{\infty}$$

$$B^{(1,m_1)} \odot B^{(m_1,m_1)^{-1}} \odot B^{(m_1,m_2)} \odot \ldots \odot B^{(m_s,1)} \Bigg) \Bigg] \cdot \hat{U}. \tag{4.17}$$

We shall now discuss the contribution of each term in eq. (4.17) to the mobility for small values of the Reynolds number. The modified Bessel functions, $I_{n+1/2}(x)$ and $K_{n+1/2}(x)$, have the following representations

$$I_{n+1/2}(x) = \left(\frac{2x}{\pi} \right)^{1/2} x^n \left(\frac{1}{x} \frac{\mathrm{d}}{\mathrm{d}x} \right)^n \frac{\sinh x}{x}, \tag{4.18}$$

$$K_{n+1/2}(x) = \left(\frac{\pi x}{2} \right)^{1/2} (-x)^n \left(\frac{1}{x} \frac{\mathrm{d}}{\mathrm{d}x} \right)^n \frac{\mathrm{e}^{-x}}{x}. \tag{4.19}$$

With these equations one finds for the behaviour of the connectors at small R

$$B^{(m,n)} = A_{m,n} R^{|m-n|}. \tag{4.20}$$

Hence each term in eq. (4.17) containing $s+1$ connectors gives a contribution to μ of order R^M, with M an even power given by

$$M = m_1 + m_s - 2 + \sum_{j=2}^{s} |m_j - m_{j-1}|, \quad m_j \neq m_{j-1}, m_j, m_{j-1} \geq 2. \tag{4.21}$$

From eq. (4.20) it follows that the expression

$$\mu = (4\pi\eta a)^{-1} \hat{U} \cdot (-B^{(1,1)} + B^{(1,2)} : B^{(2,2)^{-1}} : B^{(2,1)}) \cdot \hat{U}, \tag{4.22}$$

which contains the first two terms of the series at the right-hand side of eq. (4.17), is correct up to and including order R^3. This expression corresponds to taking into account only the zeroth and first force multipole, and neglecting all higher multipoles. We shall assume that this neglect remains justified also for large values of R.

In appendix B we show that the right-hand side of eq. (4.22) can be expressed in terms of three integrals:

$$\mu = (4\pi\eta a)^{-1}\left[\int_0^1 d\xi\,(1-\xi^2)I_{1/2}(R\xi)K_{1/2}(R\xi)\right.$$

$$+\left[\int_0^1 d\xi\,(1-\xi^2)\xi I_{3/2}(R\xi)K_{1/2}(R\xi)\right]^2$$

$$\left.\times\left[\int_0^1 d\xi\,(1-\xi^2)\xi^2 I_{3/2}(R\xi)K_{3/2}(R\xi)\right]^{-1}\right]. \tag{4.23}$$

These integrals may in turn be evaluated in a straightforward way (see appendix C) so that one has to this order in the multipole expansion:

$$\mu = (8\pi\eta aR)^{-1}\{\chi(R)+\psi(R)-\tfrac{1}{2}-[\chi(R)+2\psi(R)-1-\tfrac{2}{3}R]^2$$
$$\times[\chi(R)+\tfrac{9}{2}\psi(R)-\tfrac{7}{4}-\tfrac{1}{4}R^2-\tfrac{1}{2}e^{-2R}]^{-1}\} \tag{4.24}$$

with $\chi(R)$ and $\psi(R)$ given by:

$$\chi(R) = \gamma + \ln 2R - \mathrm{Ei}(-2R), \tag{4.25}$$

$$\psi(R) = \frac{1}{(2R)^2}(1-(1+2R)e^{-2R}). \tag{4.26}$$

In eq. (4.25), γ is Euler's constant, 0.5772 . . . , and $\mathrm{Ei}(x)$ the exponential integral

$$\mathrm{Ei}(-x) = \gamma + \ln x - \int_0^x dy\,\frac{1-e^{-y}}{y}, \quad x>0. \tag{4.27}$$

Alternatively, one may express eq. (4.23) or eq. (4.24) in terms of power series in R according to (see also appendix C)

$$\mu = (2\pi\eta a)^{-1}\left\{\sum_{n=0}^{\infty}\frac{n+2}{(n+1)(n+3)!}(-2R)^n-\left[\sum_{n=0}^{\infty}\frac{(n+1)(n+4)}{(n+3)(n+5)!}(-2R)^{n+1}\right]^2\right.$$
$$\left.\times\left[\sum_{n=0}^{\infty}\frac{(n-1)(n+2)(n+4)}{(n+3)(n+5)!}(-2R)^n\right]^{-1}\right\}. \tag{4.28}$$

With eq. (4.28) one easily shows that up to order R^3 one has for μ:

$$\mu = (6\pi\eta a)^{-1}\left[1-\frac{3}{8}R+\frac{1}{5}R^2-\frac{1}{8}R^3+\mathcal{O}(R^4)\right] \tag{4.29}$$

or equivalently

$$\mu^{-1} = 6\pi\eta a\left[1+\frac{3}{8}R-\frac{19}{320}R^2+\frac{71}{2560}R^3-\mathcal{O}(R^4)\right]. \tag{4.30}$$

This expression is identical with Goldstein's result up to order R^3 (see eq. (5.4)). It may be remarked that the coefficients in the series expansion of the mobility are much simpler than those occurring in the expansion for the friction coefficient.

Finally, to stress the similarity of the methods used in this paper for the sphere and in the previous one for the circular cylinder, we give below the expression found for the mobility of the cylinder, which is the analogue of eq. (4.23):

$$\mu = (\pi^2\eta)^{-1}\left[\int_0^1 d\xi\,(1-\xi^2)^{1/2} I_0(R\xi)K_0(R\xi) + \left[\int_0^1 d\xi\,(1-\xi^2)^{1/2}\xi I_1(R\xi)K_0(R\xi)\right]^2 \times \left[\int_0^1 d\xi\,(1-\xi^2)^{1/2}\xi^2 I_1(R\xi)K_1(R\xi)\right]^{-1}\right]. \tag{4.31}$$

In the next section we shall discuss the range of applicability of our result, eq. (4.24).

5. Discussion

In this section we shall compare for a range of Reynolds numbers the values for the drag coefficient, C_D, which follow from the results proposed by various authors. C_D is defined in terms of the mobility as

$$C_D = (\eta a R \mu)^{-1}. \tag{5.1}$$

The following expressions for this quantity are available in the literature

G.G. Stokes (1851)

$$C_D = \frac{6\pi}{R}, \tag{5.2}$$

C.W. Oseen (1913)

$$C_D = \frac{6\pi}{R}\left(1 + \frac{3}{8}R\right), \tag{5.3}$$

S. Goldstein (1929)*

$$C_D = \frac{6\pi}{R}\left(1 + \frac{3}{8}R - \frac{19}{320}R^2 + \frac{71}{2560}R^3 - \frac{30179}{2150400}R^4 + \frac{122519}{17203200}R^5\right), \tag{5.4}$$

* Corrected according to Shanks[14]).

I. Proudman et al. (1957)

$$C_D = \frac{6\pi}{R}\left(1 + \frac{3}{8}R + \frac{9}{40}R^2 \ln R\right). \tag{5.5}$$

These should be compared to the expression for C_D which follows from eq. (4.23):

$$C_D = 8\pi[\chi(R) + \psi(R) - \tfrac{1}{2} - [\chi(R) + 2\psi(R) - 1 - \tfrac{2}{3}R]^2 \times [\chi(R) + \tfrac{9}{4}\psi(R) - \tfrac{7}{4} - \tfrac{1}{4}R^2 - \tfrac{1}{2}e^{-2R}]^{-1}]^{-1}. \tag{5.6}$$

In table I we have listed for values of R in the range $R = 0.2$ to 1000 the values for C_D calculated for the five expressions given above. These results should be accompanied by a few remarks.

The values for C_D in column (5.4) of the table do not all follow from the given formula: for $R > 1$ Goldstein himself has obtained these values by numerical evaluation at an earlier stage of his solution procedure. It is seen that up to $R = 10$, values obtained from our result (listed in column (5.6)) coincide within 2% with Goldstein's. For this reason Goldstein's values have not been indicated in fig. 1. The result of Proudman and Pearson is not based solely on the Oseen equation, like all the other results, but has been obtained by matching the solution

TABLE I

This table lists values of C_D for Stokes' (5.2), Oseen's (5.3), Goldstein's (5.4), Proudman and Pearson's (5.5) and our (5.6) result and from Van Dyke's numerical evaluation (5.7)

R	(5.2)	(5.3)	(5.4)	(5.5)	(5.6)	(5.7)
0.2	94.25	101.32	101.07	99.95	101.11	
0.4	47.12	54.19	53.82	52.64	53.81	
0.6	31.42	38.48	37.95	37.18	37.96	
0.8	23.56	30.63	29.97	29.87	29.97	
1.0	18.85	25.92	25.13	25.92	25.15	
1.5	12.57	19.63	18.63	22.21	18.63	
2.0	9.42	16.49	15.30	22.37	15.30	15.30
3.0	6.28	13.35	11.88	27.33	11.87	
4.0	4.71	11.78	10.08	35.30	10.10	
5.0	3.77	10.84	8.95	44.97	8.99	9.01
10.0	1.88	8.95	6.53	106.61	6.62	6.67
15.0	1.26	8.33			5.72	5.80
20.0	0.94	8.01			5.23	5.33
25.0	0.75	7.82			4.91	5.04
50.0	0.38	7.45			4.15	4.37
100.0	0.19	7.26			3.64	
500.0	0.04	7.11			2.89	
1000.0	0.02	7.09			2.67	

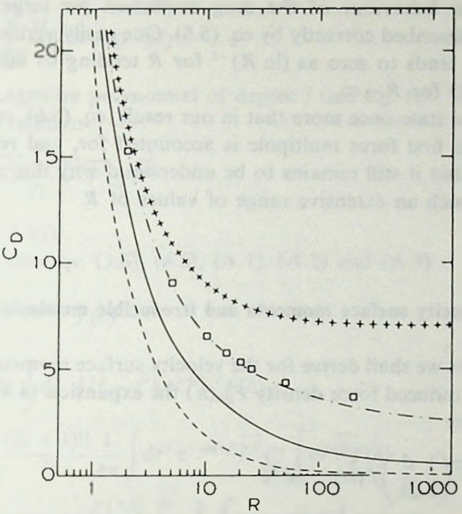

Fig. 1. C_D as function of R from experiment (——), for Stokes' (---), Oseen's (+ + +) and our (—·—) result. The results of Van Dyke are located in the centers of the small squares.

for the velocity field of the Stokes equation close to the sphere to the solution of the Oseen equation at large distances*.

The values for C_D, calculated from our result, turn out to agree remarkably well with Van Dyke's numerical data[5])** (column (5.7)), whereas those calculated according to expressions (5.3) and (5.5) differ increasingly for $R > 1$. This is also clear from fig. 1, in which C_D is plotted for all sets of values contained in table I, except those in column (5.5) which could not be plotted on the scale of the figure†. In fig. 1 we have also drawn for reference the curve corresponding to the experimental data points of Liebster and Schiller[16]) and Maxworthy[10]).

* This procedure has been carried through one further cycle by Chester and Breach in 1969[13]). Their expression is equally successful as Proudman and Pearson's in the range $0 \leqslant R \leqslant 0.5$, but yields even more unsatisfactory values for C_D for larger values of R. For this reason we have omitted this result in table I.

** For completeness' sake we mention the even more extensive numerical calculations by Payard and Bourot[17]). These yield values for C_D which are for all practical purposes equal to those obtained by Van Dyke.

† At very low Reynolds numbers, $R \leqslant 0.5$, however, Proudman and Pearson's result is in better agreement with the experiment than any result obtained from the Oseen equation beyond first order in R.

The asymptotic behaviour of the drag coefficient for large values of R is presumably not described correctly by eq. (5.6). One easily verifies that according to our result C_D tends to zero as $(\ln R)^{-1}$ for R tending to infinity. Van Dyke predicts $C_D = 3.33$ for $R \to \infty$.

To conclude we state once more that in our result, eq. (5.6), only the influence of the zeroth and first force multipole is accounted for, and remark, as in our previous paper, that it still remains to be understood why this approximation is satisfactory for such an extensive range of values of R.

Appendix A. Velocity surface moments and irreducible multipole expansion

In this appendix we shall derive for the velocity surface moments the expression (4.3) and for the induced force density $\boldsymbol{F}_{\text{ind}}(\boldsymbol{k})$ the expansion (4.4). We shall make use of the identity

$$\overline{\frac{\mathrm{d}^l}{\mathrm{d}\boldsymbol{k}^l}} f(k) = \overline{\boldsymbol{k}^l}\left(\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}k}\right)^l f(k), \quad l \in \mathbb{N} \tag{A.1}$$

with $f(k)$ an arbitrary function of $k = |\boldsymbol{k}|$. The identity may be proven by complete induction (see ref. 6). We shall also use the relation

$$J_{l+1/2}(x) = \left(\frac{2x}{\pi}\right)^{1/2}(-x)^l\left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^l \frac{\sin x}{x}. \tag{A.2}$$

By using in an appropriate way the above two equations and the definition of the velocity surface moments, eq. (4.2), one may derive eq. (4.3) in the following way

$$\begin{aligned}
\overline{\overline{\hat{\boldsymbol{r}}^p}\boldsymbol{v}(\boldsymbol{r})}^{\,\mathrm{S}} &= \frac{1}{4\pi a^2}\int \mathrm{d}\boldsymbol{r}\, \overline{\hat{\boldsymbol{r}}^p}\,\delta(r-a)\frac{1}{(2\pi)^3}\int \mathrm{d}\boldsymbol{k}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\boldsymbol{v}(\boldsymbol{k}) \\
&= \frac{1}{(2\pi)^3}\int \mathrm{d}\boldsymbol{k}\, \overline{\overline{\hat{\boldsymbol{r}}^p}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}}^{\,\mathrm{S}}\,\boldsymbol{v}(\boldsymbol{k}) \\
&= \frac{1}{(2\pi)^3}\int \mathrm{d}\boldsymbol{k}\left(-\frac{\mathrm{i}}{a}\right)^p \overline{\frac{\partial^p}{\partial\boldsymbol{k}^p}}\frac{\sin ka}{ka}\boldsymbol{v}(\boldsymbol{k}) \\
&= \frac{\mathrm{i}^p}{(2\pi)^3}\int \mathrm{d}\boldsymbol{k}\left(\frac{\pi}{2ka}\right)^{1/2} J_{p+1/2}(ka)\overline{\hat{\boldsymbol{k}}^p}\boldsymbol{v}(\boldsymbol{k}).
\end{aligned} \tag{A.3}$$

The identity

$$\delta(\hat{\boldsymbol{r}} - \hat{\boldsymbol{r}}') = \frac{1}{4\pi}\sum_{l=0}^{\infty}\frac{(2l+1)!!}{l!}\overline{\hat{\boldsymbol{r}}^l} \odot \overline{\hat{\boldsymbol{r}}'^l} \tag{A.4}$$

may be derived from a combination of the expansion

$$\delta(\hat{r}-\hat{r}') = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\hat{r}\cdot\hat{r}') \tag{A.5}$$

with $P_l(x)$ the Legendre polynomial of degree l (see e.g. ref. 18, eqs. (3.62) and (3.117)) and the relation

$$P_l(\hat{r}\cdot\hat{r}') = \frac{(2l-1)!!}{l!} \overline{\hat{r}^l} \odot \overline{\hat{r}'^l} \tag{A.6}$$

(see ref. 12, eq. (4.21)).

One obtains from eqs. (3.6), (4.5), (A.1), (A.2) and (A.4)

$$\begin{aligned}
\boldsymbol{F}_{\text{ind}}(\boldsymbol{k}) &= \int \mathrm{d}\hat{r}\, \mathrm{e}^{-\mathrm{i}a\boldsymbol{k}\cdot\hat{r}} \boldsymbol{f}(\hat{r}) \\
&= \int \mathrm{d}\hat{r} \int \mathrm{d}\hat{r}'\, \delta(\hat{r}-\hat{r}')\, \mathrm{e}^{-\mathrm{i}a\boldsymbol{k}\cdot\hat{r}} \boldsymbol{f}(\hat{r}) \\
&= \sum_{l=0}^{\infty} \frac{(2l+1)!!}{l!} \frac{1}{4\pi} \int \mathrm{d}\hat{r}'\, \mathrm{e}^{-\mathrm{i}a\boldsymbol{k}\cdot\hat{r}'} \overline{\hat{r}'^l} \odot \int \mathrm{d}\hat{r}\, \overline{\hat{r}^l} \boldsymbol{f}(\hat{r}) \\
&= \sum_{l=0}^{\infty} (2l+1)!! \left(\frac{\mathrm{i}}{a}\right)^l \left[\frac{\overline{\partial^l}}{\partial \boldsymbol{k}^l} \frac{1}{4\pi} \int \mathrm{d}\hat{r}'\, \mathrm{e}^{-\mathrm{i}a\boldsymbol{k}\cdot\hat{r}'}\right] \odot \boldsymbol{F}^{(l+1)} \\
&= \sum_{l=0}^{\infty} (2l+1)!!\, \mathrm{i}^l (ka)^l \left(\frac{1}{ka}\frac{\partial}{\partial ka}\right)^l \frac{\sin ka}{ka} \overline{\hat{k}^l} \odot \boldsymbol{F}^{(l+1)} \\
&= \sum_{l=0}^{\infty} (2l+1)!! (-\mathrm{i})^l \left(\frac{\pi}{2ka}\right)^{1/2} J_{l+1/2}(ka)\, \overline{\hat{k}^l} \odot \boldsymbol{F}^{(l+1)}
\end{aligned} \tag{A.7}$$

which is the desired expansion.

Appendix B. Derivation of eq. (4.23)

In this appendix we shall prove that eq. (4.22) may be expressed in terms of integrals according to eq. (4.23).

The connector $\boldsymbol{B}^{(1,1)}$ must be of the form

$$\boldsymbol{B}^{(1,1)} = b_1 \hat{U}\hat{U} + b_2(\mathbf{1} - \hat{U}\hat{U}) . \tag{B.1}$$

With eqs. (4.9) and (4.10) we find

$$-\hat{U}\cdot\boldsymbol{B}^{(1,1)}\cdot\hat{U} = -b_1 = \int_0^1 \mathrm{d}\xi\, (1-\xi^2) I_{1/2}(R\xi) K_{1/2}(R\xi) \tag{B.2}$$

which is the first term in eq. (4.22).

Consider now the connector $\boldsymbol{B}^{(1,2)}$. According to our convention in section 4, this tensor must be symmetric in its last two indices; in view of eq. (4.9) it is also traceless in these indices. Hence $\boldsymbol{B}^{(1,2)}$ must be of the form

$$\boldsymbol{B}^{(1,2)} = b_3 \hat U\, \overline{\hat U \hat U} + \tfrac{1}{2} b_4 (1 \hat U + \underbrace{.\hat U.}_{} - \tfrac{2}{3} \hat U 1) \qquad \text{(B.3)}$$

with $(\underbrace{.\hat U.}_{})_{\alpha\beta\gamma} \equiv \hat U_\beta \delta_{\alpha\gamma}$ and with

$$b_3 + b_4 = \tfrac{3}{2} \hat U \cdot \boldsymbol{B}^{(1,2)} : \overline{\hat U \hat U} = -\frac{9}{2} \int_0^1 \mathrm{d}\xi\, \xi (1 - \xi^2) I_{3/2}(R\xi) K_{1/2}(R\xi)\,. \qquad \text{(B.4)}$$

Since $\boldsymbol{B}^{(2a,1)}$, the part of $\boldsymbol{B}^{(2,1)}$ which is antisymmetric in its first two indices, must be proportional to $\hat U 1 - \underbrace{.\hat U.}_{}$, we see that $\boldsymbol{B}^{(2a,1)} \cdot \hat U = 0$. For $\boldsymbol{B}^{(2s,1)}$, the symmetric (and also traceless, see eq. (4.9)) part of $\boldsymbol{B}^{(2,1)}$ with respect to its first two indices, the symmetry property (4.12) yields

$$\boldsymbol{B}^{(2s,1)} = -b_3\, \overline{\hat U \hat U}\, \hat U - \tfrac{1}{2} b_4 (\hat U 1 + \underbrace{.\hat U.}_{} - \tfrac{2}{3} 1 \hat U)\,. \qquad \text{(B.5)}$$

Finally, we consider $\boldsymbol{B}^{(2s,2s)}$, which must be of the form[6,7]:

$$\boldsymbol{B}^{(2s,2s)} = b_5 \Delta + \tfrac{3}{2} b_4\, \overline{\hat U \hat U}\, \overline{\hat U \hat U} + b_7 \boldsymbol{D}(\hat U) \qquad \text{(B.6)}$$

with

$$b_5 + b_6 = \tfrac{3}{2}\, \overline{\hat U \hat U} : \boldsymbol{B}^{(2s,2s)} : \overline{\hat U \hat U} = -\frac{27}{2} \int_0^1 \mathrm{d}\xi\, \xi^2 (1 - \xi^2) I_{3/2}(R\xi) K_{3/2}(R\xi)\,. \qquad \text{(B.7)}$$

In eq. (B.6) Δ and $\boldsymbol{D}(\hat U)$ denote the standard tensors

$$\Delta = \tfrac{1}{2}(\underbrace{.\,.\,.\,.}_{} + \underbrace{.\,\underbrace{.\,.}_{}\,.}_{}) - \tfrac{1}{3} 1 1\,, \qquad \text{(B.8)}$$

$$\boldsymbol{D}(\hat U) = 2 \hat U \hat U \hat U \hat U - \tfrac{1}{2}(\hat U 1 \hat U + \underbrace{.\hat U.}_{} \hat U + \hat U \underbrace{.\hat U.}_{} + \underbrace{.\hat U \hat U.}_{})\,. \qquad \text{(B.9)}$$

Since by definition

$$\overline{\hat U \hat U} = \boldsymbol{B}^{(2s,2s)^{-1}} : \boldsymbol{B}^{(2s,2s)} : \overline{\hat U \hat U}\,, \qquad \text{(B.10)}$$

we obtain with eq. (B.6)

$$\boldsymbol{B}^{(2s,2s)^{-1}} : \overline{\hat U \hat U} = (b_5 + b_6)^{-1}\, \overline{\hat U \hat U}\,. \qquad \text{(B.11)}$$

With this last equation and eqs. (B.4) and (B.7) one easily verifies that

$$\begin{aligned}
&\hat U \cdot \boldsymbol{B}^{(1,2)} : \boldsymbol{B}^{(2,2)^{-1}} : \boldsymbol{B}^{(2,1)} \cdot \hat U \\
&= \hat U \cdot \boldsymbol{B}^{(1,2s)} : \boldsymbol{B}^{(2s,2s)^{-1}} : \boldsymbol{B}^{(2s,1)} \cdot \hat U \\
&= -(b_3 + b_4)^2\, \overline{\hat U \hat U} : \boldsymbol{B}^{(2s,2s)^{-1}} : \overline{\hat U \hat U} \\
&= -(b_3 + b_4)^2 \tfrac{2}{3} (b_5 + b_6)^{-1}
\end{aligned}$$

$$= \left[\int_0^1 d\xi\, \xi(1-\xi^2) I_{3/2}(R\xi) K_{1/2}(R\xi)\right]^2 \left[\int_0^1 d\xi\, \xi^2(1-\xi^2) I_{3/2}(R\xi) K_{3/2}(R\xi)\right]^{-1} \tag{B.12}$$

which together with eq. (B.2) leads to eq. (4.23).

Appendix C. Derivation of eqs. (4.24) and (4.25)

Consider first the integral (cf. eqs. (4.18) and (4.19))

$$I_1 = \int_0^1 d\xi\, (1-\xi^2) I_{1/2}(R\xi) K_{1/2}(R\xi) = \int_0^1 d\xi\, (1-\xi^2)\, e^{-R\xi} \frac{\sinh R\xi}{R\xi}. \tag{C.1}$$

Defining $2R\xi \equiv y$, I_1 takes the form (cf. ref. 13, eq. (8.212.1))

$$I_1 = \frac{1}{2R}\int_0^{2R} dy \frac{1-e^{-y}}{y} - \frac{1}{(2R)^3}\int_0^{2R} dy\, y(1-e^{-y})$$

$$= \frac{1}{2R}(\gamma + \ln 2R - \mathrm{Ei}(-2R)) + \frac{1}{(2R)^3}(1 - 2R^2 - (1+2R)\, e^{-2R}). \tag{C.2}$$

Using the standard series representation of the function $\gamma + \ln x - \mathrm{Ei}(-x)$(cf. ref. 13, eq. (8.214.1)) the integral I_1 may also be written as

$$I_1 = -\frac{1}{2R}\sum_{n=1}^{\infty} \frac{(-2R)^n}{n \cdot n!} + \frac{1}{(2R)^3}\left(1 - 2R^2 - (1+2R)\sum_{n=0}^{\infty}\frac{(-2R)^n}{n!}\right)$$

$$= 2\sum_{n=0}^{\infty} \frac{(n+2)}{(n+1)(n+3)!}(-2R)^n. \tag{C.3}$$

Consider now the integral

$$I_2 = \int_0^1 d\xi\, (1-\xi^2)\xi I_{3/2}(R\xi) K_{1/2}(R\xi)$$

$$= \int_0^1 d\xi\, (1-\xi^2)\xi\, e^{-R\xi} \frac{d}{dR\xi}\frac{\sinh R\xi}{R\xi}. \tag{C.4}$$

After partial integration and the substitution $2R\xi = y$, I_2 becomes

$$I_2=\frac{1}{(2R)^2}\int_0^{2R}\mathrm{d}y\left(1-\frac{2}{y}\right)(1-\mathrm{e}^{-y})-\frac{1}{(2R)^4}\int_0^{2R}\mathrm{d}y\,(y^2-6y)(1-\mathrm{e}^{-y})$$

$$=-\frac{1}{2R^2}\left(\gamma+\ln 2R-1-\frac{2}{3}R-\mathrm{Ei}(-2R)\right)-\frac{1}{4R^4}(1-(1+2R)\,\mathrm{e}^{-2R})\,. \tag{C.5}$$

In a similar way as was done above for I_1, we find for I_2 the series representation

$$I_2=-2\sum_{n=0}^{\infty}\frac{(n+1)(n+4)}{(n+3)(n+5)!}(-2R)^{n+1}\,. \tag{C.6}$$

Finally, we turn to the integral

$$I_3=\int_0^1\mathrm{d}\xi\,(1-\xi^2)\xi^2 I_{3/2}(R\xi)K_{3/2}(R\xi)$$

$$=-\int_0^1\mathrm{d}\xi\,(1-\xi^2)\xi^2\left[R\xi\frac{\mathrm{d}}{\mathrm{d}R\xi}\frac{\mathrm{e}^{-R\xi}}{R\xi}\right]\frac{\mathrm{d}}{\mathrm{d}R\xi}\frac{\sinh R\xi}{R\xi}\,. \tag{C.7}$$

After partial integration I_3 becomes

$$I_3=\frac{1}{(2R)^3}\int_0^{2R}\mathrm{d}y\left(y-2-\frac{4}{y}\right)(1-\mathrm{e}^{-y})$$

$$-\frac{1}{(2R)^5}\int_0^{2R}\mathrm{d}y\,(y^3-6y^2-12y)(1-\mathrm{e}^{-y})$$

$$=-\frac{1}{2R^3}\left(\gamma+\ln 2R-\frac{7}{4}-\frac{1}{4}R^2-\mathrm{Ei}(-2R)\right)$$

$$-\frac{1}{16R^5}(9-(4R^2+18R+9)\,\mathrm{e}^{-2R})\,. \tag{C.8}$$

It then follows that I_3 has the series representation

$$I_3=-2\sum_{n=0}^{\infty}\frac{(n-1)(n+2)(n+4)}{(n+3)(n+5)!}(-2R)^n\,. \tag{C.9}$$

Collecting the results (C.2), (C.5) and (C.8) we obtain eq. (4.24); with eqs. (C.3), (C.6) and (C.9) one obtains eq. (4.25).

Note added in proof

The authors are indebted to Prof. R. Dorfman for drawing their attention to the article: Stokes Problems in Kinetic-Theory by G. Scharf (Phys. Fluids 13 (1970) 848) who obtains using the linearized Boltzmann equation, an expression for the drag equivalent, in the case of the cylinder, to our lowest order result. This illustrates the close relationship between hydrodynamics and kinetic theory, and between the Oseen equation and the linearized Boltzmann equation in particular.

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Note added in proof

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Chapter III

Drag on a sphere moving slowly in a rotating viscous fluid

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Drag on a sphere moving slowly in a rotating viscous fluid

1. Introduction

The drag force experienced by a sphere that moves along the axis of a rotating incompressible viscous fluid depends in a quite complicated way on the velocity U of the sphere and the angular velocity Ω of the fluid far away from the sphere. This conclusion may be drawn from experiments carried out by Maxworthy (1965, 1970). The present-day theoretical insight in this phenomenon is rather limited, since the fluid equations are solvable only after drastic simplification.

Among the first to consider motion in a rotating fluid were Proudman (1916), Taylor (1922) and Grace (1926). The latter obtained a formula for the ultimate drag on a sphere, started impulsively in an inviscid incompressible fluid, albeit with an estimated numerical coefficient. Many years later, Stewartson (1952) derived the exact expression.

Morrison & Morgan (1956) and Moore & Saffman (1969), among others, included viscosity, but considered steady motion in a rapidly rotating fluid. Their result for the drag on a sphere is identical with Stewartson's. Childress (1964) studied the motion of a sphere in a viscous fluid in a different regime, in which both the Taylor number $T \equiv \rho\Omega a^2/\eta$ and the Reynolds number $R \equiv \rho U a/\eta$ are small (a is the radius of the sphere; ρ and η are the density and viscosity of the fluid). He was able to determine a first correction to the Stokes drag, proportional to $T^{\frac{1}{2}}$. Recently Dennis, Ingham & Singh (1982) solved the fluid equations of motion numerically, and calculated the drag for $T \leqslant 0.5$ and $R \leqslant 0.5$. In all treatments mentioned above explicit solutions were constructed for the velocity field and pressure field in the entire fluid. Subsequently the drag was calculated by integration of the normal component of the pressure tensor over the surface of the sphere.

In this paper we shall evaluate the friction assuming that all momentum convection in the fluid may be neglected, i.e. for zero Reynolds number. We do not impose any restriction on the value of the Taylor number, so that the results also cover a range not considered before. Our analysis makes use of a method of induced forces, which was developed by Mazur & van Saarloos (1982) to analyse many-sphere hydrodynamic interactions in Stokes flow, and was applied by Mazur and the present author (Mazur

& Weisenborn 1984; Weisenborn & Mazur 1984) to evaluate the Oseen drag on a circular cylinder and a sphere. This method evades the need of constructing explicit solutions for the fluid fields.

In §2 we briefly discuss the equations of motion and continuity for the fluid.

In §3 we introduce an induced force density on the sphere in the equation of motion and give the formal solution for the velocity field in wavevector representation, in terms of this induced force density.

In §4 we expand the induced force density in irreducible force multipoles. Applying the boundary condition to the so-called velocity surface moments, we derive a set of coupled linear equations for the force multipoles. We then use this set to obtain a formal expression for the translational mobility, the inverse of the friction. With this expression we may evaluate the transitional mobility on the basis of all multipoles up to an arbitrary order, while neglecting the contributions of all multipoles of higher order. The actual evaluation, performed on the basis of the first five multipoles, shows that the multipole expansion converges rapidly: values for the mobility, based on the first three force-multipoles, are for all Taylor numbers altered by less than 1 % if the influence of the fifth force-multipole is also accounted for. We further derive an alternative expression which enables us to calculate the first seven terms in the expansion of the translational mobility in powers of $T^{\frac{1}{4}}$. Finally we give the corresponding expression for the rotational mobility for the case where the applied torque is parallel to the rotation axis of the fluid, and we calculate with this expression the first three terms of the expansion of the rotational mobility in powers of $T^{\frac{1}{4}}$.

A discussion of the results is given in §5. This discussion includes a comparison with theoretical results obtained previously as well as to experimental data where available.

2. The equations of motion and continuity

We consider a sphere of radius a that moves with constant velocity $\boldsymbol{U}$ along the axis of a rotating fluid. If unperturbed by the presence of the sphere, this viscous, imcompressible and unbounded fluid rotates uniformly with constant angular velocity $\boldsymbol{\Omega}$. In a non-rotating frame of reference the equation of motion (the Navier–Stokes equation) and the equation of continuity read

$$\left.\begin{aligned} \rho\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}(\boldsymbol{r},t) &= -\nabla\cdot\boldsymbol{P}(\boldsymbol{r},t), \qquad (2.1)\\ \nabla\cdot\boldsymbol{v}(\boldsymbol{r},t) &= 0, \qquad (2.2)\end{aligned}\right\} \quad \text{for } |\boldsymbol{r}-\boldsymbol{R}(t)|>a,$$

with

$$P_{\alpha\beta} = p\delta_{\alpha\beta}-\eta\left(\frac{\partial v_\alpha}{\partial r_\beta}+\frac{\partial v_\beta}{\partial r_\alpha}\right). \qquad (2.3)$$

Here $\mathrm{d}/\mathrm{d}t$ is the substantial time derivative, $\boldsymbol{v}$ the velocity field, $\boldsymbol{P}$ the pressure tensor, p the hydrostatic pressure, and η and ρ the viscosity and density of the fluid. $\boldsymbol{R}(t)$ denotes the position of the centre of the sphere.

Upon transformation to a frame of reference that corotates with the unperturbed fluid, (2.1), combined with (2.3), becomes

$$\rho\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}(\boldsymbol{r},t)+2\rho\boldsymbol{\Omega}\wedge\boldsymbol{v}(\boldsymbol{r},t) = -\nabla p^*(\boldsymbol{r},t)+\eta\,\Delta\boldsymbol{v}(\boldsymbol{r},t) \quad \text{for } |\boldsymbol{r}-\boldsymbol{R}(t)|>a. \qquad (2.4)$$

The reduced hydrostatic pressure $p^*(\boldsymbol{r}, t)$ is defined as

$$p^*(\boldsymbol{r},t) \equiv p(\boldsymbol{r},t) - \tfrac{1}{2}\rho(\Omega^2 r^2 - (\boldsymbol{\Omega}\cdot\boldsymbol{r})^2). \tag{2.5}$$

In (2.4) and (2.5), d/dt, $\boldsymbol{v}$ and $\boldsymbol{r}$ denote the substantial time derivative, the velocity field and the position vector with respect to the rotating frame; $\Omega \equiv |\boldsymbol{\Omega}|$.

We now choose the origin of the rotating coordinate frame at the centre of the moving sphere. The fluid motion then becomes time-independent and obeys the equation

$$\rho(\boldsymbol{v}(\boldsymbol{r}) - \boldsymbol{U})\cdot\nabla\boldsymbol{v}(\boldsymbol{r}) + 2\rho\boldsymbol{\Omega}\wedge\boldsymbol{v}(\boldsymbol{r}) = -\nabla p^*(\boldsymbol{r}) + \eta\Delta\boldsymbol{v}(\boldsymbol{r}) \quad \text{for } r > a.$$

Full linearization of this equation with respect to the velocities of both the fluid and the sphere amounts to neglecting the first term on its left-hand side. We shall assume that both velocities are small enough to justify this linearization. The equation of motion now becomes

$$2\rho\boldsymbol{\Omega}\wedge\boldsymbol{v}(\boldsymbol{r}) = -\nabla p^*(\boldsymbol{r}) + \eta\Delta\boldsymbol{v}(\boldsymbol{r}) \quad \text{for } r > a. \tag{2.6}$$

This equation must be supplemented by appropriate boundary conditions at the surface of the sphere. We choose stick boundary conditions:

$$\boldsymbol{v}(\boldsymbol{r}) = \boldsymbol{U} + \boldsymbol{\omega}\wedge\boldsymbol{r} \quad \text{for } r = a, \tag{2.7}$$

with $\boldsymbol{\omega}$ the angular velocity of the sphere in the rotating coordinate frame. We consider only the case where the sphere experiences a torque in the direction of the angular velocity $\boldsymbol{\Omega}$. For symmetry reasons the vector $\boldsymbol{\omega}$ must then be parallel to $\boldsymbol{\Omega}$, i.e. $\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\hat{\boldsymbol{\Omega}})\hat{\boldsymbol{\Omega}}$ with $\hat{\boldsymbol{\Omega}} \equiv \boldsymbol{\Omega}/\Omega$.

3. Formulation of the problem using induced forces

The concept of induced forces enables us to formulate the problem, posed by (2.2), (2.6) and (2.7), in an alternative way, as follows: we extend the fluid equations within the sphere and write them in the form

$$\left.\begin{aligned} 2\rho\boldsymbol{\Omega}\wedge\boldsymbol{v}(\boldsymbol{r}) &= -\nabla p^*(\boldsymbol{r}) + \eta\Delta\boldsymbol{v}(\boldsymbol{r}) + \boldsymbol{F}_{\text{ind}}(\boldsymbol{r}), \\ \nabla\cdot\boldsymbol{v}(\boldsymbol{r}) &= 0 \end{aligned}\right\} \text{ for all } \boldsymbol{r}, \quad \begin{aligned}(3.1)\\(3.2)\end{aligned}$$

with $\boldsymbol{F}_{\text{ind}}(\boldsymbol{r}) = 0$ for $r > a$. As extension of the velocity field within the sphere, we choose

$$\boldsymbol{v}(\boldsymbol{r}) = \boldsymbol{U} + \boldsymbol{\omega}\wedge\boldsymbol{r} \quad \text{for } r \leqslant a. \tag{3.3}$$

On the reduced hydrostatic pressure we impose the condition

$$p^*(\boldsymbol{r}) = \rho(\boldsymbol{\Omega}\cdot\boldsymbol{\omega})\,\boldsymbol{r}\cdot(\boldsymbol{I} - \hat{\boldsymbol{\Omega}}\hat{\boldsymbol{\Omega}})\cdot\boldsymbol{r} \quad \text{for } r < a. \tag{3.4}$$

This implies, in view of (2.5), for the extension of the hydrostatic pressure

$$p(\boldsymbol{r}) = \rho\boldsymbol{\Omega}\cdot(\boldsymbol{\omega} + \tfrac{1}{2}\boldsymbol{\Omega})\,\boldsymbol{r}\cdot(\boldsymbol{I} - \hat{\boldsymbol{\Omega}}\hat{\boldsymbol{\Omega}})\cdot\boldsymbol{r} \quad \text{for } r < a. \tag{3.5}$$

The extensions in (3.4) and (3.5) do not include $r = a$, since the stick boundary condition (2.7) uniquely determines the pressure on the surface of the sphere. In (3.4) and (3.5) $\boldsymbol{I}$ denotes the unit tensor. From substitution of (3.3) and (3.4) in (3.1) it follows that the induced force density $\boldsymbol{F}_{\text{ind}}(\boldsymbol{r})$ is of the form

$$\boldsymbol{F}_{\text{ind}}(\boldsymbol{r}) = a^{-2}\boldsymbol{f}(\hat{\boldsymbol{r}})\,\delta(r - a). \tag{3.6}$$

The factor a^{-2} has been introduced here for convenience, $\hat{\boldsymbol{r}} \equiv \boldsymbol{r}/r$.

If we make use of (2.3), (2.5), (3.1) and (3.3), as well as of Gauss' theorem, we can express the force $\boldsymbol{K}$ exerted by the fluid on the sphere in terms of the induced force density according to

$$\boldsymbol{K} = -\int_{r=a} \mathrm{d}S\,\hat{\boldsymbol{r}}\cdot\boldsymbol{P}(\boldsymbol{r}) = -\int_{r\leqslant a} \mathrm{d}\boldsymbol{r}\,\nabla\cdot\boldsymbol{P}(\boldsymbol{r}) = -\int \mathrm{d}\boldsymbol{r}\,\boldsymbol{F}_{\mathrm{ind}}(\boldsymbol{r}). \tag{3.7}$$

In a similar way we may also relate the torque $\boldsymbol{T}$ that the fluid exerts on the sphere to the induced force density. We have

$$\boldsymbol{T} = -\int_{r=a} \mathrm{d}S\,\boldsymbol{r}\wedge(\hat{\boldsymbol{r}}\cdot\boldsymbol{P}(\boldsymbol{r})) = -\int_{r\leqslant a} \mathrm{d}\boldsymbol{r}\,\boldsymbol{r}\wedge(\nabla\cdot\boldsymbol{P}(\boldsymbol{r})) = -\int \mathrm{d}\boldsymbol{r}\,\boldsymbol{r}\wedge\boldsymbol{F}_{\mathrm{ind}}(\boldsymbol{r}). \tag{3.8}$$

In order to solve formally the equation of motion for the fluid we introduce Fourier transforms; for example, the velocity field:

$$\boldsymbol{v}(\boldsymbol{k}) \equiv \int \mathrm{d}\boldsymbol{r}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\,\boldsymbol{v}(\boldsymbol{r}).$$

Equation (3.1) and (3.2) become in wavevector representation

$$(\eta k^2 + 2\rho\boldsymbol{\Omega}\wedge)\,\boldsymbol{v}(\boldsymbol{k}) = -\mathrm{i}\boldsymbol{k}p^*(\boldsymbol{k}) + \boldsymbol{F}_{\mathrm{ind}}(\boldsymbol{k}), \tag{3.9}$$

$$\boldsymbol{k}\cdot\boldsymbol{v}(\boldsymbol{k}) = 0. \tag{3.10}$$

We now apply the operator $\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}$, where $\hat{\boldsymbol{k}} \equiv \boldsymbol{k}/k$, to both sides of (3.9) and make use of (3.10). We then obtain the equation

$$\eta(k^2 + 2Ta^{-2}(\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\cdot[\hat{\boldsymbol{\Omega}}\wedge(\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})]\cdot)\,\boldsymbol{v}(\boldsymbol{k}) = (\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\cdot\boldsymbol{F}_{\mathrm{ind}}(\boldsymbol{k}). \tag{3.11}$$

Here T is the Taylor number defined as

$$T \equiv \frac{\rho\Omega a^2}{\eta}.$$

The tensor $(\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\cdot[\hat{\boldsymbol{\Omega}}\wedge(\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})]$ acts on an arbitrary vector $\boldsymbol{s}$ in the following way:

$$\begin{aligned}(\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\cdot[\hat{\boldsymbol{\Omega}}\wedge(\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})]\cdot\boldsymbol{s} &= -(\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\cdot[\hat{\boldsymbol{\Omega}}\wedge(\hat{\boldsymbol{k}}\wedge(\hat{\boldsymbol{k}}\wedge\boldsymbol{s}))]\\ &= (\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\cdot[(\hat{\boldsymbol{\Omega}}\cdot\hat{\boldsymbol{k}})\,\hat{\boldsymbol{k}}\wedge\boldsymbol{s} - (\hat{\boldsymbol{\Omega}}\cdot(\hat{\boldsymbol{k}}\wedge\boldsymbol{s}))\,\hat{\boldsymbol{k}}]\\ &= (\hat{\boldsymbol{\Omega}}\cdot\hat{\boldsymbol{k}})\,\hat{\boldsymbol{k}}\wedge\boldsymbol{s} = -\xi\hat{\boldsymbol{k}}\cdot\boldsymbol{\epsilon}\cdot\boldsymbol{s}.\end{aligned} \tag{3.12}$$

Here $\boldsymbol{\epsilon}$ is the Levi-Civita tensor and $\xi \equiv \hat{\boldsymbol{\Omega}}\cdot\hat{\boldsymbol{k}}$. With (3.12) the equation of motion (3.11) can be written as

$$\eta k^2\left(\boldsymbol{I} - \frac{2T\xi}{k^2a^2}\,\boldsymbol{k}\cdot\boldsymbol{\epsilon}\right)\cdot\boldsymbol{v}(\boldsymbol{k}) = (\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\cdot\boldsymbol{F}_{\mathrm{ind}}(\boldsymbol{k}).$$

One easily verifies, using (3.10), that the formal solution of this equation is given by

$$\boldsymbol{v}(\boldsymbol{k}) = \frac{k^2a^4}{\eta(k^4a^4 + 4T^2\xi^2)}\left(\boldsymbol{I} - \hat{\boldsymbol{k}}\hat{\boldsymbol{k}} + \frac{2T\xi}{k^2a^2}\,\boldsymbol{k}\cdot\boldsymbol{\epsilon}\right)\cdot\boldsymbol{F}_{\mathrm{ind}}(\boldsymbol{k}). \tag{3.13}$$

This solution implies that the unperturbed fluid is at rest in the rotating frame of reference.

4. Evaluation of the mobility

The aim of our analysis is the evaluation of the translational mobility of the sphere for arbitrary values of the Taylor number. We show in §4.1 that for the problem under consideration translation and rotation of the sphere do not couple. Since the tensors relating U to K and ω to T have a trivial form, owing to the symmetry of the problem, both the translational mobility μ^{T} and the rotational mobility μ^{R} may be defined as scalar quantities:

$$U \equiv -\mu^{\mathrm{T}} K, \tag{4.1}$$

$$\omega \equiv -\mu^{\mathrm{R}} T. \tag{4.2}$$

For the translational mobility we derive in §§4.2 and 4.3 two different formal expressions. One of these is suitable for numerical evaluation of μ^{T}; the other expression enables us to obtain by an in principle simple calculation the first seven terms in the expansion of μ^{T} in powers of $T^{\frac{1}{2}}$. For the rotational mobility we derive a formal expression in section 4.3 and give the first three terms of its expansion in powers of $T^{\frac{1}{2}}$.

4.1. *A hierarchy of equations for the force multipoles*

As first step in the evaluation of μ^{T} we expand the induced force in irreducible force multipoles and derive for these multipoles a hierarchy of equations.

For the induced force density $F_{\mathrm{ind}}(k)$ we may use the following expansion (see Appendix A)

$$F_{\mathrm{ind}}(k) = \sum_{l=0}^{\infty} (2l+1)!!\,(-i)^l j_l(ka)\,\overline{\hat{k}^l} \odot F^{(l+1)} \tag{4.3}$$

with

$$F^{(l+1)} \equiv (l!)^{-1} \int \mathrm{d}\hat{r}\, \overline{\hat{r}^l} f(\hat{r}). \tag{4.4}$$

$F^{(l+1)}$ is the $(l+1)$th irreducible force multipole moment;

$$(2l+1)!! \equiv (2l+1)(2l-1) \ldots \times 5 \times 3;$$

$j_l(ka)$ is the spherical Bessel function of order l with argument ka. $\overline{\hat{k}^l}$ denotes the irreducible, i.e. symmetric and traceless, tensor of rank l constructed with the vector $\hat{k}$ (see, for example, Hess & Koehler 1980, §1.1). The symbol $\odot$ denotes the full, in this case l-fold, contraction of the tensors $\overline{\hat{k}^l}$ and $F^{(l+1)}$, with the convention that the last index of $\overline{\hat{k}^l}$ is contracted with the first index of $F^{(l+1)}$, etc.

According to (3.6), (3.7) and (4.4), the force K is related to the first force multipole:

$$K = -F^{(1)}. \tag{4.5}$$

Similarly it follows from (3.6), (3.8) and (4.4) that

$$T = a\epsilon : F^{(2)}. \tag{4.6}$$

In order to derive a hierarchy of equations for the force multipoles, we shall determine the so-called velocity surface moments, defined as

$$\overline{\overline{\hat{r}^p}\, v(r)}^{S} \equiv (4\pi a^2)^{-1} \int \mathrm{d}r\, \overline{\hat{r}^p}\, v(r)\, \delta(r-a), \quad p = 0, 1, 2, \ldots. \tag{4.7}$$

It may be shown that the following identity holds (see Appendix A):

$$\overline{\overline{\hat{r}^p}\, v(r)}^{S} = \frac{\mathrm{i}^p}{(2\pi)^3} \int \mathrm{d}k\, j_p(ka)\, \overline{\hat{k}^p}\, v(k), \quad p = 0, 1, 2, \ldots. \tag{4.8}$$

We now substitute (4.3) into (3.13), and the resulting equation into the right-hand side of (4.8). We evaluate the left-hand side of (4.8) with the boundary condition (2.7).

The result of this procedure is the desired hierarchy of equations:

$$U\delta_{p0}+a\boldsymbol{\epsilon}\cdot\boldsymbol{\omega}\delta_{p1} = (4\pi\eta a)^{-1}\sum_{l=0}^{\infty}(1-2\delta_{pl})\mathscr{B}^{(p+1,\,l+1)}\odot \boldsymbol{F}^{(l+1)},\quad p=0,1,2,\ldots. \tag{4.9}$$

The factor $1-2\delta_{pl}$ is introduced for convenience. The tensors $\mathscr{B}^{(p+1,\,l+1)}$ of rank $p+l+2$, which will be called *connectors*, are given by

$$\mathscr{B}^{(p+1,\,l+1)} = (\boldsymbol{B}_1^{(p+1,\,l+1)}+\boldsymbol{B}_2^{(p+1,\,l+1)})(1-2\delta_{pl}), \tag{4.10}$$

with

$$\boldsymbol{B}_1^{(p+1,\,l+1)} = (2p+1)!!\,(2l+1)!!\frac{1}{4\pi}\int \mathrm{d}\hat{\boldsymbol{k}}\,\overline{\hat{\boldsymbol{k}}^{p}}\,(\boldsymbol{I}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\,\overline{\hat{\boldsymbol{k}}^{l}}\;\mathrm{Re}\,B^{(p+1,\,l+1)}, \tag{4.11}$$

$$\boldsymbol{B}_2^{(p+1,\,l+1)} = (2p+1)!!\,(2l+1)!!\frac{1}{4\pi}\int \mathrm{d}\hat{\boldsymbol{k}}\,\overline{\hat{\boldsymbol{k}}^{p}}\,(\hat{\boldsymbol{k}}\cdot\boldsymbol{\epsilon})\,\overline{\hat{\boldsymbol{k}}^{l}}\;\mathrm{Im}\,B^{(p+1,\,l+1)}, \tag{4.12}$$

and with

$$B^{(p+1,\,l+1)} = \mathrm{i}^{p-l}\frac{2}{\pi}\int_0^\infty \mathrm{d}x\,\frac{x^2 j_p(x)\,j_l(x)}{x^4+4T^2\xi^2}(x^2+2\mathrm{i}T\xi). \tag{4.13}$$

In (4.11) and (4.12) Re and Im denote the real and imaginary part; in (4.13) $x \equiv ka$.

It is easily checked that $\mathscr{B}^{(p+1,\,l+1)}$ vanishes if $p+l$ is odd, owing to the integration over $\hat{\boldsymbol{k}}$ in (4.11) and (4.12). This allows the separation of hierarchy (4.9) into two sets of equations, namely

$$U\delta_{p0} = (4\pi\eta a)^{-1}\sum_{l=0}^{\infty}(1-2\delta_{pl})\mathscr{B}^{(2p+1,\,2l+1)}\odot \boldsymbol{F}^{(2l+1)},\quad p=0,1,2,\ldots, \tag{4.14}$$

and

$$a\boldsymbol{\epsilon}\cdot\boldsymbol{\omega}\delta_{p1} = (4\pi\eta a)^{-1}\sum_{l=1}^{\infty}(1-2\delta_{pl})\mathscr{B}^{(2p,\,2l)}\odot \boldsymbol{F}^{(2l)},\quad p=1,2,3,\ldots. \tag{4.15}$$

The above decomposition ensures that translation and rotation do not couple, as is required by symmetry for zero Reynolds number.

From (4.11) and (4.12) it follows that the two connector parts $\boldsymbol{B}_1^{(p+1,\,l+1)}$ and $\boldsymbol{B}_2^{(p+1,\,l+1)}$ satisfy the symmetry relations

$$\boldsymbol{B}_1^{(p+1,\,l+1)} = (-1)^{p+l}\,\widetilde{\boldsymbol{B}_1^{(l+1,\,p+1)}}, \tag{4.16a}$$

$$\boldsymbol{B}_2^{(p+1,\,l+1)} = (-1)^{p+l+1}\,\widetilde{\boldsymbol{B}_2^{(l+1,\,p+1)}}. \tag{4.16b}$$

Here $\tilde{\boldsymbol{C}}$ denotes the generalized transposed of an arbitrary tensor $\boldsymbol{C}$ of rank q, defined as

$$(\tilde{\boldsymbol{C}})_{\alpha_1,\alpha_2,\ldots,\alpha_q} \equiv (\boldsymbol{C})_{\alpha_q,\ldots,\alpha_2,\alpha_1}.$$

We now evaluate the scalar quantity $B^{(p+1,\,l+1)}$. Considering only the case where $p+l$ is even, the integration over $\hat{\boldsymbol{k}}$ in (4.11) and (4.12) may be replaced by twice the integration over those $\hat{\boldsymbol{k}}$ for which $\xi = \boldsymbol{\Omega}\cdot\hat{\boldsymbol{k}} \geqslant 0$. In Appendix B it is shown how one can obtain by means of complex integration for $B^{(p+1,\,l+1)}$ the expression

$$B^{(p+1,\,l+1)} = \mathrm{i}^{p+1-l}\,z\,j_{\max(p,\,l)}(z)\,h^{(1)}_{\min(p,\,l)}(z), \tag{4.17a}$$

with

$$z = (1+\mathrm{i})(T\xi)^{\frac{1}{2}}. \tag{4.17b}$$

In (4.17) $\max(p,l)$ and $\min(p,l)$ denote respectively the larger and smaller integer of the pair p and l. $h_n^{(1)}(z)$ is the first spherical Bessel function of the third kind of order n with argument z (see, for example, Abramowitz & Stegun 1968). For small values of their arguments, $j_n(z)$ and $h_n^{(1)}(z)$ may be expanded in ascending power series in z:

$$j_n(z) = \frac{z^n}{(2n+1)!!} + O(z^{n+2}),$$

$$h_n^{(1)}(z) = -\mathrm{i}\frac{(2n-1)!!}{z^{n+1}} + O(z^{-n}).$$

One may check by substituting these formulae in (4.17) and the resulting expressions for $B^{(p+1,l+1)}$ in (4.11) and (4.12) that the connectors behave for small values of T as

$$\mathscr{B}^{(p+1,l+1)} = \mathscr{A}_{p,l}\, T^{\frac{1}{2}|p-l|} + O(T^{\frac{1}{2}(|p-l|+1)}). \tag{4.18}$$

4.2. *Systematic evaluation of the translational mobility*

We shall now derive from the set of equations (4.14) an expression for the translational mobility. This expression will enable us to evaluate μ^{T} on the basis of all multipoles up to a certain order, while neglecting the contributions from higher multipoles.

First we shall construct the explicit tensorial form of the irreducible force multipoles and velocity surface moments. By definition the irreducible force multipoles are integrals of the tensor $\overline{\hat{r}^l} f(\hat{r})$ over the unit sphere (cf. (4.4)). According to Hess & Koehler (1980, equation (2.50)), this tensor can for $l \geqslant 1$ be split up into three parts as follows:

$$\overline{\hat{r}^l} f(\hat{r}) = \Delta^{(l+1)} \odot f(\hat{r})\,\overline{\hat{r}^l} - \frac{l(2l-1)}{l(2l+1)-1}\,\square^{(l)} \odot (f(\hat{r}) \wedge \overline{\hat{r}^l}) + \frac{2l-1}{2l+1}\,\Delta^{(l)} \odot (f(\hat{r}) \cdot \overline{\hat{r}^l}). \tag{4.19}$$

Here $\Delta^{(l)}$ is a tensor of rank $2l$ that projects out the irreducible part of a tensor of rank l, while $\square^{(l)}$ is defined as

$$(\square^{(l)})_{\mu_1,\ldots,\mu_l,\lambda,\mu_1',\ldots,\mu_l'} \equiv (\Delta^{(l)})_{\mu_1,\ldots,\mu_l,\nu_1,\ldots,\nu_{l-1},\nu_l} (\epsilon)_{\nu_l,\lambda,\nu_l'} (\Delta^{(l)})_{\nu_l',\nu_1,\ldots,\nu_{l-1},\mu_1',\ldots,\mu_l'}.$$

The decomposition (4.19) represents a generalization of the standard decomposition of a tensor of rank two into its traceless symmetric and antisymmetric parts and its trace.

Since the connectors are tensors constructed solely with the unit vector $\hat{\Omega}$ (cf. (4.10)–(4.12)), it follows from (4.14) that all force multipoles $F^{(2l+1)}$ are linear in $\hat{U}$. Upon integration over $\hat{r}$, the tensors on the right-hand side of (4.19) therefore each contain one unit vector $\hat{U}$ and the appropriate number of unit vectors $\hat{\Omega}$. Hence we may write

$$F^{(2l+1)} = \sum_{i=1}^{3} F_i^{(2l+1)}\, \boldsymbol{a}_i^{(2l+1)}, \tag{4.20}$$

with

$$\left.\begin{aligned} \boldsymbol{a}_1^{(2l+1)} &\equiv \left(\frac{(4l+1)!!}{(2l+1)!}\right)^{\frac{1}{2}} \Delta^{(2l+1)} \odot \hat{U}\hat{\Omega}^{2l}, \\ \boldsymbol{a}_2^{(2l+1)} &\equiv \left(\frac{(4l-1)!!\,2l}{(2l+1)!}\right)^{\frac{1}{2}} \square^{(2l)} \odot \hat{U}\hat{\Omega}^{2l-1}, \\ \boldsymbol{a}_3^{(2l+1)} &\equiv \left(\frac{(4l-1)!!}{(2l-1)!\,(4l+1)}\right)^{\frac{1}{2}} \Delta^{(2l)} \odot \hat{U}\hat{\Omega}^{2l-2} \end{aligned}\right\} \quad (l \geqslant 1), \tag{4.21a}$$

and for $l = 0$

$$\boldsymbol{a}_1^{(1)} \equiv \hat{U}, \quad \boldsymbol{a}_2^{(1)} \equiv 0, \quad \boldsymbol{a}_3^{(1)} \equiv 0. \tag{4.21b}$$

In Appendix C we show that for all $l \geqslant 1$ the tensors $\boldsymbol{a}_i^{(2l+1)}$ satisfy the relation

$$\widetilde{\boldsymbol{a}_i^{(2l+1)}} \odot \boldsymbol{a}_j^{(2l+1)} \equiv (-1)^{i+1}\,\delta_{ij}, \quad i,j = 1,2,3. \tag{4.22}$$

In view of the identical structure of (4.4) and (4.7), the velocity surface moments can be split up in an analogous way.

We now introduce the scalar quantities $b_{i,j}^{(2p+1,\,2l+1)}$, defined as

$$b_{i,j}^{(2p+1,\,2l+1)} \equiv (1-2\delta_{pl})\,\widetilde{\boldsymbol{a}_i^{(2p+1)}} \odot \mathscr{B}^{(2p+1,\,2l+1)} \odot \boldsymbol{a}_j^{(2l+1)}, \quad i,j = 1,2,3. \tag{4.23}$$

These quantities satisfy the symmetry relation (cf. (4.10), (4.16a,b) and (4.21))

$$b_{i,j}^{(2p+1,\,2l+1)} = b_{j,i}^{(2l+1,\,2p+1)}. \tag{4.24}$$

Using (4.20) and (4.23), we can write (4.14) in the form

$$4\pi\eta a U \delta_{p0} = \sum_{l=0}^{\infty}\sum_{j=1}^{3} b_{i,j}^{(2p+1,\,2l+1)}\,F_j^{(2l+1)}, \quad i = 1,2,3;\ p = 0,1,2,\ldots. \tag{4.25}$$

If we truncate this set of equations at $p = l = M$, we can solve $\boldsymbol{K} = \boldsymbol{F}^{(1)} = -\hat{\boldsymbol{U}}F_1^{(1)}$ from the remaining finite set of equations by application of Cramer's rule. This yields an expression for the translational mobility which takes into account the influence of the first $M+1$ multipoles with odd superscripts:

$$\mu^{\mathrm{T}}(M) = (4\pi\eta a)^{-1}\begin{cases} b_{1,1}^{(1,1)} & (M=0), \\ |\boldsymbol{b}(M)|\,|\boldsymbol{b}'(M)|^{-1} & (M \geqslant 1), \end{cases} \tag{4.26}$$

where $|\boldsymbol{b}(M)|$ is the determinant of the matrix $\boldsymbol{b}(M)$ with elements $b_{i,j}^{(2\alpha+1,\,2\beta+1)}$, $\alpha,\beta = 0,1,2,\ldots,M$; $i,j = 1,2,3$, and $|\boldsymbol{b}'(M)|$ the determinant of the matrix $\boldsymbol{b}'(M)$ with elements $b_{i,j}^{(2\gamma+1,\,2\delta+1)}$, $\gamma,\delta = 1,2,\ldots,M$; $i,j = 1,2,3$. The true mobility is obtained in the limit $M \to \infty$.

For $M = 0$ the expression for the translational mobility reads explicitly

$$\mu^{\mathrm{T}}(0) = -(4\pi\eta a)^{-1}\,\hat{\boldsymbol{U}}\cdot\mathscr{B}^{(1,1)}\cdot\hat{\boldsymbol{U}}. \tag{4.27}$$

Using (4.10), (4.11) and (4.17), as well as the relations

$$j_0(z) = \frac{\sin z}{z}, \quad h_0^{(1)}(z) = -\frac{\mathrm{i}}{z}\mathrm{e}^{\mathrm{i}z},$$

we may evaluate $\hat{\boldsymbol{U}}\cdot\mathscr{B}^{(1,1)}\cdot\hat{\boldsymbol{U}}$ as follows:

$$\begin{aligned}\hat{\boldsymbol{U}}\cdot\mathscr{B}^{(1,1)}\cdot\hat{\boldsymbol{U}} &= -\int_0^1 \mathrm{d}\xi\,(1-\xi^2)\,\mathrm{Re}\,B^{(1,1)} \\ &= -\tfrac{2}{5}T^{-\frac{1}{2}} - \tfrac{1}{32}T^{-3}\,\mathrm{e}^{-2T^{\frac{1}{2}}}\big((8T^{\frac{3}{2}} - 6T^{\frac{1}{2}} - 3)\sin 2T^{\frac{1}{2}} + (8T^{\frac{3}{2}} + 12T + 6T^{\frac{1}{2}})\cos 2T^{\frac{1}{2}}\big).\end{aligned} \tag{4.28}$$

Combining (4.27) and (4.28), we obtain an expression for μ^{T} that contains the contribution of the first force multipole alone. Its behaviour for large values of T is easily seen to be

$$\mu^{\mathrm{T}}(0) = (10\pi\eta a)^{-1}\,T^{-\frac{1}{2}}.$$

In Appendix E it is shown that $\mu^{\mathrm{T}}(1)$ behaves asymptotically as

$$\mu^{\mathrm{T}}(1) = \tfrac{3}{16}(\eta a T)^{-1}\{1 + O(T^{-\frac{1}{2}})\}. \tag{4.29}$$

In the limit $\eta \to 0$ the above expression for $\mu^{\mathrm{T}}(1)$ is equivalent to Stewartson's formula for the ultimate drag experienced by an impulsively started sphere moving in a rotating inviscid fluid (see (5.2)). We have not been able to prove that the same result for the mobility is obtained in the limit $M \to \infty$, i.e. for the true mobility. We note, however, that Stewartson's result has also been derived for the drag on a sphere moving steadily in a rapidly rotating viscous fluid (see, for example, Moore & Saffman 1969), and therefore we feel justified to presume that for T tending to infinity we would indeed find

$$\mu^{\mathrm{T}} = \mu^{\mathrm{T}}(\infty) = \tfrac{3}{16}(\eta a T)^{-1}.$$

Equation (4.29) then implies that the first and third force-multipole moments together determine the asymptotic behaviour of the translational mobility.

In order to estimate the influence of the inclusion of higher multipoles on μ^{T} for arbitrary values of the Taylor number, we have evaluated $\mu^{\mathrm{T}}(M)$ numerically for $M = 0, 1$ and 2. The results of the calculations can be found in §5; they show that the first and third force-multipoles have a dominating influence on the mobility for *all* values of the Taylor number.

4.3. *Power-series expansion for the translational mobility*

In this subsection we shall derive for μ^{T} an alternative expression that is in particular convenient for analysing the behaviour of μ^{T} for small values of T. We shall also give the corresponding expression for μ^{R}.

Using (4.5), we rewrite the set of equations (4.14) as follows:

$$4\pi\eta a \boldsymbol{U} = \mathscr{B}^{(1,1)}\cdot \boldsymbol{K} + \sum_{l=1}^{\infty} \mathscr{B}^{(1,2l+1)} \odot \boldsymbol{F}^{(2l+1)}, \tag{4.30}$$

$$\boldsymbol{F}^{(2l+1)} = \mathscr{B}^{(2l+1,2l+1)^{-1}} \odot \Big(-\mathscr{B}^{(2l+1,1)}\cdot \boldsymbol{K} + \sum_{\substack{m=1\\ m\neq l}}^{\infty} \mathscr{B}^{(2l+1,2m+1)} \odot \boldsymbol{F}^{(2m+1)}\Big) \quad (l \geqslant 1). \tag{4.31}$$

Here $\mathscr{B}^{(n,n)^{-1}}$ denotes the inverse of $\mathscr{B}^{(n,n)}$, defined only if $\mathscr{B}^{(n,n)}$ acts on a tensor of rank n that is irreducible in its first $n-1$ indices. By iteration we can eliminate all higher multipoles from the right-hand side of (4.31) in favour of $\boldsymbol{K}$. When the resulting equations are substituted in (4.30) we get an equation of the form (4.1), yielding for μ^{T} the expression

$$\mu^{\mathrm{T}} = (4\pi\eta a)^{-1}\, \hat{U}\cdot\Big(-\mathscr{B}^{(1,1)} + \sum_{s=1}^{\infty} \Big[\sum_{m_1=1}^{\infty} \cdots \sum_{\substack{m_s=1\\ m_s\neq m_{s-1}}}^{\infty} \Big] \times \mathscr{B}^{(1,2m_1+1)} \odot \mathscr{B}^{(2m_1+1,2m_1+1)^{-1}} \odot \cdots \odot \mathscr{B}^{(2m_s+1,2m_s+1)^{-1}} \odot \mathscr{B}^{(2m_s+1,1)}\Big)\cdot \hat{U}. \tag{4.32}$$

From comparison of (4.26) and (4.32), it follows that for $M \geqslant 2$ the expression for $\mu^{\mathrm{T}}(M)$ given in (4.26) corresponds to a partial resummation of (4.32), involving an infinite number of terms, each of which is a part of the direct contribution of the first $M+1$ force multipoles with odd superscripts to μ^{T}. The expression corresponding to $\mu^{\mathrm{T}}(1)$ is given by

$$\mu^{\mathrm{T}} = (4\pi\eta a)^{-1}\, \hat{U}\cdot\big(-\mathscr{B}^{(1,1)} + \mathscr{B}^{(1,3)} \odot \mathscr{B}^{(3,3)^{-1}} \odot \mathscr{B}^{(3,1)}\big)\cdot \hat{U}. \tag{4.33}$$

With the help of (4.18), we can easily check that the expansion of this expression in powers of $T^{\frac{1}{2}}$ will be correct up to $T^{\frac{7}{2}}$, since contributions from higher multipoles than the quadrupole are at least of order T^4. In Appendix F it is shown that this expansion is given by

$$\mu^{\mathrm{T}} = (6\pi\eta a)^{-1}\big(1 - \tfrac{4}{7}T^{\frac{1}{2}} + \tfrac{8}{45}T^{\frac{3}{2}} - \tfrac{4}{45}T^2 - \tfrac{485504}{2029052025}T^3 + \tfrac{32}{1053}T^{\frac{7}{2}}\big) + O(T^4). \tag{4.34}$$

We note that this series does not contain terms of order T and $T^{5/2}$, and also that the coefficient of the term proportional to T^3 is much smaller than the coefficients of the other terms. Indeed, if we wish to compute values for μ^{T} using (4.36) for comparison with experimental data we may as well neglect this term (cf. table 1 in §5).

We now turn to the other set of equations, the set (4.15), which relates the angular velocity $\boldsymbol{\omega}$ to the force multipoles with even superscripts. We may derive from this set a relation between $\boldsymbol{\omega}$ and T of the form (4.2), yielding for μ^{R} the expression

$$\mu^{\mathrm{R}} = (16\pi\eta a^3)^{-1}\,\hat{\boldsymbol{\omega}}\cdot\boldsymbol{\epsilon}:\Bigg(-\mathscr{B}^{(2,2)} + \sum_{s=1}^{\infty}\Bigg[\sum_{m_1=1}^{\infty}{}' \cdots \sum_{\substack{m_s=1 \\ m_s \neq m_{s-1}}}^{\infty}{}'\Bigg]$$

$$\times\, \mathscr{B}^{(2,2m_1)} \odot \mathscr{B}^{(2m_1,2m_1)^{-1}} \odot \cdots \odot \mathscr{B}^{(2m_s,2m_s)^{-1}} \odot \mathscr{B}^{(2m_s,2)}\Bigg):\boldsymbol{\epsilon}\cdot\hat{\boldsymbol{\omega}}.$$

Here $\Sigma'^{\infty}_{m_i=1}$ denotes a summation over all integer values $m_i \geqslant 1$ $(i = 1, 2, \ldots, s)$ with the proviso that for $m_i = 1$ only the part of $\mathscr{B}^{(\ ,2m_i)}$ or $\mathscr{B}^{(2m_i,\ )}$ symmetric in the corresponding two indices is included in the summation. Expansion of the above expression for μ^{R} up to $T^{3/2}$ yields

$$\mu^{\mathrm{R}} = (8\pi\eta a^3)^{-1}\,(1 - \tfrac{8}{45}T^{3/2}) + O(T^2).$$

We note that terms proportional to $T^{1/2}$ and T are absent in this expansion.

5. Discussion

When comparing experimental with theoretical results for the drag force experienced by a slowly moving spherical particle, it is convenient to introduce a dimensionless drag, defined as

$$\frac{D}{D_{\mathrm{s}}} \equiv \frac{D}{6\pi\eta a U} = (6\pi\eta a\mu^{\mathrm{T}})^{-1}.$$

The following two expressions for this quantity are available in the literature for the problem under consideration:

Childress (1964) $$\frac{D}{D_{\mathrm{s}}} = 1 + \tfrac{4}{7}T^{1/2} \quad \text{for } T \ll 1, \tag{5.1}$$

Stewartson (1952) $$\frac{D}{D_{\mathrm{s}}} = \frac{8}{9\pi}T \quad \text{for } T \rightarrow \infty. \tag{5.2}$$

Childress's result is equivalent to the first term in our expansion of μ^{T} in powers of $T^{1/2}$, (4.34); Stewartson's result has already been discussed in §4.2. We shall now compare values for D/D_{s} calculated using the above expressions with experimentally and numerically obtained values for this quantity, as well as with values calculated from (4.26) for $\mu^{\mathrm{T}}(0)$, $\mu^{\mathrm{T}}(1)$ and $\mu^{\mathrm{T}}(2)$.

We first consider Taylor numbers in the range from zero to unity. Column (1) of table 1 contains values for $(D/D_{\mathrm{s}}) - 1$ calculated with (5.1). Values for this quantity calculated using the first seven terms in the series expansion for μ^{T} are listed in column (2). The results obtained by Dennis *et al.* (1982) from a numerical solution of the full Navier–Stokes equation at $R = 0.12$ are listed in column (3). Column (4) contains values for $(D/D_{\mathrm{s}}) - 1$ calculated using (4.28). These values are, to within the given accuracy, identical with those obtained from (4.26) for $M = 1$ and $M = 2$. Values for $(D/D_{\mathrm{s}}) - 1$ obtained by extrapolation of Maxworthy's (1965) experimental data are listed in column 5, together with the errors caused by this extrapolation.

T	(1)	(2)	(3)	(4)	(5)
0.025	0.09	0.10	0.107	0.10	0.09 ± 0.01
0.050	0.13	0.14	—	0.14	0.13 ± 0.02
0.075	0.16	0.18	—	0.18	0.18 ± 0.02
0.10	0.18	0.21	—	0.21	0.22 ± 0.03
0.20	0.26	0.32	—	0.32	0.30 ± 0.04
0.25	0.29	0.37	0.398	0.37	0.37 ± 0.04
0.50	0.40	0.56	0.631	0.57	0.57 ± 0.05
0.75	0.49	0.72	—	0.74	0.75 ± 0.05
1.0	0.57	0.83	—	0.90	—

TABLE 1. Values for $(D/D_s) - 1$, calculated using Childress' result (1), the series expansion (2) and the monopole approximation (4); column 3 giving the results of Dennis *et al.*, and column 5 the experimental values

T	(1)	(2)	(3)
1.0	19.0	19.0	19.0
2.5	10.8	10.8	10.8
5.0	7.47	7.74	7.74
7.5	6.08	6.58	6.58
10.0	5.27	5.94	5.94
25.0	3.33	4.61	4.60
50.0	2.36	4.02	4.02
75.0	1.92	3.78	3.77
100.0	1.67	3.64	3.64
250.0	1.05	3.33	3.32
500.0	7.45×10^{-1}	3.17	3.17
750.0	6.09×10^{-1}	3.11	3.11
1000.0	5.27×10^{-1}	3.07	3.07
10000.0	1.67×10^{-1}	2.90	2.90
100000.0	5.27×10^{-2}	2.85	2.85

TABLE 2. Values for $10D/D_s T$, computed on the basis of the first (1), the first and third (2) and the first, third and fifth (3) force multipole moments

It is seen that the results computed using Childress' formula (5.1) are only satisfactory up to $T = 0.2$, whereas those computed using (4.34) are still satisfactory at $T = 0.75$, when compared with the experimental data. The results listed in column (4), which were calculated using the monopole approximation of the induced force, compare very favourably with those in column (5), whereas the numerical results of Dennis *et al.*, column (3), do not agree too well with the experimental values. One should, however, keep in mind that they were calculated for a very small but finite Reynolds number, and must be corrected accordingly.

We have listed in table 2 values for $D/D_s T$ for Taylor numbers above unity. Column 1 contains results computed with the monopole approximation, while in columns (2) and (3) the influences of respectively the third and of the third and fifth force-multipole moments is taken into account. It is seen that values obtained from the monopole approximation deviate by less than 15 % from those obtained from the monopole + quadrupole approximation for $T \leqslant 10.0$. The inclusion of the influence of the hexadecapole moment does not alter significantly the values for $D/D_s T$ obtained from the monopole + quadrupole approximation for *any* value of the Taylor number. As anticipated, Stewartson's limiting value $8/9\pi$ for $D/D_s T$ is approached more and more closely for higher and higher values of T in columns 2 and 3.

Maxworthy (1970) has determined experimentally that D/D_s behaves for large values of T as

$$\frac{D}{D_s} = (0.43 \pm 0.01) T^{1.00 \pm 0.01}.$$

In an attempt to explain the discrepancy between this result and Stewartson's expression (5.2), Hocking, Moore & Walton (1979) have analysed the influence of the finite axial size of the container, used by Maxworthy for his experiments, on the value of the drag. The results of their analysis suggest this is not the main cause of the discrepancy. To determine whether (and, if so, to what extent) nonlinear effects, in particular the influence of momentum convection, are responsible for the discrepancy, it would be interesting to compare the values of $D/D_s T$ for Taylor numbers above unity, listed in column (3) of table 2, with experimentally obtained values for this quantity. Except for the asymptotic value, quoted above, such experimental results have not been found in the presently available literature.

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Appendix A. Irreducible force multipoles and velocity surface moments

In this appendix we shall derive for the induced force density $\boldsymbol{F}_{ind}(\boldsymbol{k})$ the expansion (4.3), and for the velocity surface moments the expression (4.8). We shall make use of the following identity:

$$\frac{\overline{\mathrm{d}^l}}{\mathrm{d}\boldsymbol{k}^l} f(k) = \overline{\hat{\boldsymbol{k}}^l}\, k^l \left(\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}k}\right)^l f(k), \quad l \in \mathbb{N}$$

with $f(k)$ an arbitrary function of $k = |\boldsymbol{k}|$. For the case where $f(k) = j_0(ka)$, this identity becomes, using also Rayleigh's formula (see Abramowitz & Stegun 1968),

$$\frac{\overline{\mathrm{d}^l}}{\mathrm{d}\boldsymbol{k}^l} j_0(ka) = (-a)^l\, \overline{\hat{\boldsymbol{k}}^l}\, j_l(ka). \tag{A 1}$$

We shall also use the identity

$$\delta(\hat{\boldsymbol{r}} - \hat{\boldsymbol{r}}') = \frac{1}{4\pi} \sum_{l=0}^{\infty} \frac{(2l+1)!!}{l!}\, \overline{\hat{\boldsymbol{r}}^l} \odot \overline{\hat{\boldsymbol{r}}'^l}, \tag{A 2}$$

which may be derived by combination of the expansion

$$\delta(\hat{\boldsymbol{r}} - \hat{\boldsymbol{r}}') = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}')$$

with $P_l(x)$ the Legendre polynomial of degree l (see, for example, Jackson 1975, equations (3.62) and (3.117)), and the relation

$$P_l(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}') = \frac{(2l-1)!!}{l!}\, \overline{\hat{\boldsymbol{r}}^l} \odot \overline{\hat{\boldsymbol{r}}'^l}$$

(see Hess & Koehler 1980, equation (4.21)).

We obtain from (3.6), (4.4), (A 1) and (A 2):

$$
\begin{aligned}
F_{\text{ind}}(k) &= \int d\hat{\rho}\, e^{-iak\cdot\hat{\rho}} f(\hat{\rho}) \\
&= \int d\hat{\rho} \int d\hat{\rho}'\, \delta(\hat{\rho}-\hat{\rho}')\, e^{-iak\cdot\hat{\rho}'} f(\hat{\rho}') \\
&= \sum_{l=0}^{\infty} \frac{(2l+1)!!}{l!} \frac{1}{4\pi} \int d\hat{\rho}'\, e^{-iak\cdot\hat{\rho}'}\, \overline{\hat{\rho}'^{l}} \odot \int d\hat{\rho}\, \overline{\hat{\rho}^{l}} f(\hat{\rho}) \\
&= \sum_{l=0}^{\infty} (2l+1)!! \left(\frac{i}{a}\right)^{l} \left[\overline{\frac{\partial^{l}}{\partial k^{l}}} \frac{1}{4\pi} \int d\hat{\rho}'\, e^{-iak\cdot\hat{\rho}'}\right] \odot F^{(l+1)} \\
&= \sum_{l=0}^{\infty} (2l+1)!!\, (-i)^{l} j_{l}(ka)\, \overline{\hat{k}^{l}} \odot F^{(l+1)},
\end{aligned}
$$

which is the desired expansion. Combining (4.7) and (A 1), we may derive (4.8) in the following way:

$$
\begin{aligned}
\overline{\overline{\hat{\rho}^{p}}\, v(r)}^{S} &= \frac{1}{4\pi a^{2}} \int dr\, \overline{\hat{\rho}^{p}}\, \delta(r-a) \frac{1}{(2\pi)^{3}} \int dk\, e^{ik\cdot r}\, v(k) \\
&= \frac{1}{(2\pi)^{3}} \int dk \left[\frac{1}{4\pi} \int d\hat{\rho}\, e^{iak\cdot\hat{\rho}}\, \overline{\hat{\rho}^{p}}\right] v(k) \\
&= \frac{i^{p}}{(2\pi)^{3}} \int dk\, j_{p}(ka)\, \overline{\hat{k}^{p}}\, v(k).
\end{aligned}
$$

Appendix B. Evaluation of $B^{(p+1,\ell+1)}$

In this appendix we shall evaluate the quantity $B^{(p+1,\ell+1)}$, given by (see eq. (4.13))

$$
B^{(p+1,\ell+1)} = \frac{i^{p-\ell}}{\pi} \int_{-\infty}^{\infty} dx\, \frac{x^{2}\, j_{p}(x)\, j_{\ell}(x)}{x^{2} - 2iT\xi} \qquad (\ \xi > 0\ ,\ p+\ell \text{ even}\ )
$$

$$
= \frac{i^{p-\ell}}{\pi} \int_{-\infty}^{\infty} dx\, \frac{x\, j_{p}(x)\, j_{\ell}(x)}{x-z}
$$

with $z = (1+i)(T\xi)^{\frac{1}{2}}$, by means of complex integration. With the help of the notation of section 4.2 we can replace this integral by

$$
\lim_{\epsilon\to 0} \frac{i^{p-\ell}}{\pi} \int_{-\infty}^{\infty} dx\, \frac{x\, j_{\max(p,\ell)}(x)\, j_{\min(p,\ell)}(x[1+\epsilon])}{x-z} .
$$

Since the spherical Bessel functions $j_n(x)$ satisfy for all values of n an inequality of the form

$$|j_n(x)| < \frac{\text{constant}}{1 + |x|}$$

it follows from the principle of dominated convergence (see e.g. Feller (1971)) that the outcome of the integration does not depend on the way in which the limit is taken (i.e. $\varepsilon \uparrow 0$ or $\varepsilon \downarrow 0$). Using the relation (see Abramowitz and Stegun (1968))

$$j_n(x) = \tfrac{1}{2} \left(h_n^{(1)}(x) + h_n^{(2)}(x) \right) ,$$

we obtain for $B^{(p+1,\ell+1)}$:

$$B^{(p+1,\ell+1)} = \lim_{\varepsilon \downarrow 0} \frac{i^{p-\ell}}{2\pi} \int_{-\infty}^{\infty} dx \; \frac{x \, j_{\max(p,\ell)}(x) \, h^{(1)}_{\min(p,\ell)}(x[1+\varepsilon])}{x-z}$$

$$+ \lim_{\varepsilon \downarrow 0} \frac{i^{p-\ell}}{2\pi} \int_{-\infty}^{\infty} dx \; \frac{x \, j_{\max(p,\ell)}(x) \, h^{(2)}_{\min(p,\ell)}(x[1+\varepsilon])}{x-z} \quad .$$

To evaluate the first integral we use as contour a large semicircle above the real axis with its centre at the origin, together with that part of the real axis which joins the ends of the semicircle; for the second integral we use the reflection of this contour with respect to the real axis. The integrals round the large semicircles both tend to zero as the radii tend to infinity due to the exponential decay of the functions $h_m^{(1)}(x)$ and $h_m^{(2)}(x)$. Applying now Cauchy's residue theorem we find that the second integral vanishes and that the first integral yields the expression for $B^{(p+1,\ell+1)}$ given in eq. (4.17).

Appendix C. Proof of eq. (4.22)

In this appendix we shall show that the tensors $\underline{a}_i^{(2\ell+1)}$, defined in eq.

(4.21) for $\ell > 1$, satisfy the relation

$$\overbrace{\underline{a}_i^{(2\ell+1)}} \odot \underline{a}_j^{(2\ell+1)} = (-1)^{i+1}\,\delta_{ij} \quad , \quad i,j = 1,2,3 \quad .$$

Many of the relations we shall use in this appendix can be found in chapter 2 of Hess and Koehler (1980). With the help of the relation

$$\overline{\hat{b}^{2\ell+1}} = \overline{\hat{b}^{2\ell}}\;\hat{b} - \frac{2\ell}{4\ell+1}\,\underline{\Delta}^{(2\ell)} \odot \overline{\hat{b}^{2\ell-1}}$$

and the identities

$$\hat{b}^{2\ell+1} \odot \underline{\Delta}^{(2\ell+1)} = \overline{\hat{b}^{2\ell+1}} \odot \underline{\Delta}^{(2\ell+1)} = \hat{b}\;\overline{\hat{b}^{2\ell}} \odot \underline{\Delta}^{(2\ell+1)}$$

one easily verifies that

$$\begin{aligned}
\overline{\hat{b}^{2\ell+1}} \odot \overline{\hat{b}^{2\ell+1}} &= \hat{b}\;\overline{\hat{b}^{2\ell}} \odot \overline{\hat{b}^{2\ell+1}} \\
&= \hat{b}\;\overline{\hat{b}^{2\ell}} \odot \left(\overline{\hat{b}^{2\ell}}\;\hat{b} - \frac{2\ell}{4\ell+1}\,\underline{\Delta}^{(2\ell)} \odot \overline{\hat{b}^{2\ell-1}} \right) \\
&= \frac{2\ell+1}{4\ell+1}\,\overline{\hat{b}^{2\ell}} \odot \overline{\hat{b}^{2\ell}} \\
&= \frac{(2\ell+1)!}{(4\ell+1)!!} \quad .
\end{aligned} \qquad \text{(C.1)}$$

Using eq. (C.1) and the fact that $(\hat{\Omega}\cdot\hat{U})^2 = 1$, one finds

$$\overbrace{\underline{a}_1^{(2\ell+1)}} \odot \underline{a}_1^{(2\ell+1)} = \frac{(4\ell+1)!!}{(2\ell+1)!}\;\hat{\Omega}^{2\ell}\,\hat{U} \odot \underline{\Delta}^{(2\ell+1)} \odot \underline{\Delta}^{(2\ell+1)} \odot \hat{U}\,\hat{\Omega}^{2\ell}$$

$$= \frac{(4\ell+1)!!}{(2\ell+1)!}\;\overline{\hat{\Omega}^{2\ell}\,\hat{U}} \odot \overline{\hat{U}\,\hat{\Omega}^{2\ell}} = 1 \quad .$$

Combining the identity

$$\underline{\square}^{(2\ell)} \odot^{*} \underline{\square}^{(2\ell)} = -\,\frac{2\ell+1}{2\ell}\,\underline{\Delta}^{(2\ell)} \quad ,$$

where $\odot^{*}$ denotes the $2\ell + 1$ fold contraction, with eq. (C.1) it will be clear that

$$\overset{\frown}{\underline{a}_2^{(2\ell+1)}} \odot \underline{a}_2^{(2\ell+1)} = \frac{(4\ell+1)!!\,2\ell}{(2\ell+1)!} \; \hat{\Omega}^{2\ell-1}\, \hat{U} \odot \underline{\Box}^{(2\ell)} \odot^* \underline{\Box}^{(2\ell)} \odot \hat{U}\, \hat{\Omega}^{2\ell-1}$$

$$= -1 \quad .$$

Finally, using the relation

$$(\underline{\Delta}^{(2\ell)})_{\mu_1,..,\mu_{2\ell-1},\lambda,\mu_1',..,\mu_{2\ell-1}',\lambda} = \frac{4\ell+1}{4\ell-1} (\underline{\Delta}^{(2\ell-1)})_{\mu_1,..,\mu_{2\ell-1},\mu_1',..,\mu_{2\ell-1}'}$$

and eq. (C.1) one may check that

$$\overset{\frown}{\underline{a}_3^{(2\ell+1)}} \odot \underline{a}_3^{(2\ell+1)} = \frac{(4\ell-1)!!}{(2\ell-1)!(4\ell+1)} \; \hat{\Omega}^{2\ell-2}\, \hat{U} \odot \underline{\Delta}^{(2\ell)} \odot^* \underline{\Delta}^{(2\ell)} \odot \hat{U}\, \hat{\Omega}^{2\ell-2}$$

$$= 1 \quad .$$

Appendix D

As preparation for appendices E and F we evaluate in this appendix the quantities $b_{i,j}^{(2p+1,2\ell+1)}$ for the cases $p,\ell = 0,1$ and $i,j = 1,2,3$.

Using eq. (4.17) and the relations (see e.g. Abramowitz and Stegun (1968))

$$j_{2n}(z) = z^{2n} \left(\frac{1}{z}\frac{d}{dz} \right)^{2n} \frac{\sin z}{z} \quad , \quad h_{2n}^{(1)}(z) = z^{2n} \left(\frac{1}{z}\frac{d}{dz} \right)^{2n} \frac{e^{iz}}{iz} \quad ,$$

one has, denoting $2(T\xi)^{\frac{1}{2}}$ by y, for $B^{(1,3)}$ and $B^{(3,3)}$:

$$B^{(1,3)} = \frac{1}{2y} \left(1 - \frac{6}{y^2} + (1 + \frac{6}{y} + \frac{6}{y^2})\, e^{-y} \sin y \;-\; (1 - \frac{6}{y^2})\, e^{-y} \cos y\right)$$

$$+ \frac{i}{2y} \left(1 - \frac{6}{y} + \frac{6}{y^2} - (1 - \frac{6}{y^2})\, e^{-y} \sin y \;-\; (1 + \frac{6}{y} + \frac{6}{y^2})\, e^{-y} \cos y\right)$$

$$= \frac{1}{60} y^3 - \frac{1}{105} y^4 + \frac{1}{504} y^5 - i\left(\frac{1}{30} y^2 - \frac{1}{60} y^3 + \frac{1}{504} y^5\right) + O(y^6) \quad ,$$

$$B^{(3,3)} = \frac{1}{2y} \left(1 + \frac{6}{y^2} - \frac{36}{y^4} + (1 + \frac{12}{y} + \frac{30}{y^2} - \frac{36}{y^4})\, e^{-y} \sin y\right.$$

$$\left. + (-1 + \frac{30}{y^2} + \frac{72}{y^3} + \frac{36}{y^4})\, e^{-y} \cos y\right)$$

$$+ \frac{1}{2y} \left(1 - \frac{6}{y^2} - \frac{36}{y^4} - (1 - \frac{30}{y^2} - \frac{72}{y^3} - \frac{36}{y^4}) e^{-y} \sin y\right.$$

$$\left. - (1 + \frac{12}{y} + \frac{30}{y^2} - \frac{36}{y^4}) e^{-y} \cos y\right]$$

$$= \frac{1}{5} + \frac{1}{105} y^2 + O(y^4) \quad .$$

In view of the symmetry relation (4.24) we have to evaluate $b_{i,j}^{(2p+1,2\ell+1)}$ only for $p < \ell$, while in the case $p = \ell$ only for $i > j$. For the cases $p,\ell = 0,1$ the contractions in eq. (4.23) can be carried out very easily. Below we have listed expressions for $b_{i,j}^{(2p+1,2\ell+1)}$ for $p,\ell = 0,1$ and $i,j = 1,2,3$ for small and large values of T. In the expressions for large values of T terms proportional to $\exp(-2\surd T)$ have been left out of consideration.

$$b_{1,1}^{(1,1)} = \int_0^1 d\xi \, (1 - \xi^2) \,\mathrm{Re}\, B^{(1,1)}$$

$$= \frac{2}{3} (1 - \frac{4}{7} T^{\frac{1}{2}} + \frac{8}{45} T^{3/2} - \frac{8}{75} T^2 + \frac{32}{1155} T^{5/2} - \frac{32}{12285} T^{7/2}) + O(T^4)$$

$$= \frac{2}{5} T^{-\frac{1}{2}} \quad ,$$

$$b_{1,1}^{(1,3)} = \frac{3}{2} \surd 10 \int_0^1 d\xi \, (1 - \xi^2)(1 - 5\xi^2) \,\mathrm{Re}\, B^{(1,3)}$$

$$= \frac{32}{975} \surd 10 \, T^{3/2} (1 - \frac{52}{49} T^{\frac{1}{2}}) + O(T^{5/2})$$

$$= - \frac{4}{15} \surd 10 \, T^{-\frac{1}{2}} (1 - \frac{135}{64} \pi T^{-\frac{1}{2}}) + O(T^{-3/2}) \quad ,$$

$$b_{1,2}^{(1,3)} = 15 \int_0^1 d\xi \, (1 - \xi^2) \, \xi \,\mathrm{Im}\, B^{(1,3)}$$

$$= \frac{4}{15} T (1 - \frac{60}{77} T^{\frac{1}{2}} + \frac{200}{819} T^{3/2}) + O(T^3)$$

$$= - \frac{10}{7} T^{-\frac{1}{2}} (1 - \frac{21}{4} T^{-\frac{1}{2}}) + O(T^{-3/2}) \quad ,$$

$$b_{1,3}^{(1,3)} = \surd 15 \int_0^1 d\xi \, (1 - \xi^2) \,\mathrm{Re}\, B^{(1,3)}$$

$$= - \frac{16}{675} \sqrt{15}\, T^{3/2} \left(1 - \frac{6}{7} T^{\frac{1}{2}} \right) + O(T^{5/2})$$

$$= - \frac{2}{5} \sqrt{15}\, T^{-\frac{1}{2}} \left(1 - \frac{15}{16} \pi\, T^{-\frac{1}{2}} \right) + O(T^{-3/2}) \quad ,$$

$$b_{1,1}^{(3,3)} = \frac{45}{2} \int_0^1 d\xi \left(1 - \xi^2 \right) \left(1 - 5\xi^2 \right)^2 \operatorname{Re} B^{(3,3)}$$

$$= \frac{24}{7} + O(T^2)$$

$$= \frac{112}{13} T^{-\frac{1}{2}} + O(T^{-3/2}) \quad ,$$

$$b_{1,2}^{(3,3)} = \frac{45}{2} \sqrt{10} \int_0^1 d\xi \left(1 - \xi^2 \right) \left(1 - 5\xi^2 \right) \xi \operatorname{Im} B^{(3,3)}$$

$$= - \frac{32}{245} \sqrt{10}\, T + O(T^2)$$

$$= - \frac{60}{77} \sqrt{10}\, T^{-\frac{1}{2}} + O(T^{-3/2}) \quad ,$$

$$b_{1,3}^{(3,3)} = \frac{15}{2} \sqrt{6} \int_0^1 d\xi \left(1 - \xi^2 \right) \left(1 - 5\xi^2 \right) \operatorname{Re} B^{(3,3)}$$

$$= O(T^2)$$

$$= \frac{4}{3} \sqrt{6}\, T^{-\frac{1}{2}} + O(T^{-3/2}) \quad ,$$

$$b_{2,2}^{(3,3)} = - 225 \int_0^1 d\xi \left(1 - \xi^2 \right) \xi^2 \operatorname{Re} B^{(3,3)}$$

$$= - 6 + O(T^2)$$

$$= - 10\, T^{-\frac{1}{2}} + O(T^{-3/2}) \quad ,$$

$$b_{2,3}^{(3,3)} = 15 \sqrt{15} \int_0^1 d\xi \left(1 - \xi^2 \right) \xi \operatorname{Im} B^{(3,3)}$$

$$= \frac{8}{105} \sqrt{15}\, T + O(T^2)$$

$$= \frac{10}{7}\sqrt{15}\, T^{-\frac{1}{2}} + O(T^{-3/2}) \quad ,$$

$$b_{3,3}^{(3,3)} = 15 \int_0^1 d\xi \, (\, 1 - \xi^2 \,) \, \mathrm{Re}\, B^{(3,3)}$$

$$= 2 + O(T^2)$$

$$= 6\, T^{-\frac{1}{2}} + O(T^{-3/2}) \quad .$$

Appendix E. Asymptotic behaviour of $\mu^T(1)$

In this appendix we show that the asymptotic behaviour of $\mu^T(1)$ for large values of the Taylor number is given by

$$\mu^T(1) = \frac{3}{16} \left(\eta a T\right)^{-1} \left(\, 1 + O(T^{-\frac{1}{2}}) \,\right) \quad . \tag{E.1}$$

After substituting in eq. (4.26) the asymptotic expressions for $b_{i,j}^{(2p+1,2\ell+1)}$ for $p,\ell = 0,1$; $i,j = 1,2,3$, evaluated in appendix D, and carrying out some standard manipulations one obtains for $\mu^T(1)$

$$\mu^T(1) = (4\pi\eta a)^{-1} \begin{vmatrix} \frac{112}{13} T^{-\frac{1}{2}} & -\frac{60}{77}\sqrt{10}\, T^{-\frac{1}{2}} & \frac{4}{3}\sqrt{6}\, T^{-\frac{1}{2}} \\ -\frac{60}{77}\sqrt{10}\, T^{-\frac{1}{2}} & -10\, T^{-\frac{1}{2}} & \frac{10}{7}\sqrt{15}\, T^{-\frac{1}{2}} \\ \frac{4}{3}\sqrt{6}\, T^{-\frac{1}{2}} & \frac{10}{7}\sqrt{15}\, T^{-\frac{1}{2}} & 6\, T^{-\frac{1}{2}} \end{vmatrix}^{-1}$$

$$\times \begin{vmatrix} \frac{6\pi}{8} T^{-1} & \frac{9\pi}{16}\sqrt{10}\, T^{-1} & \frac{15}{2} T^{-1} & \frac{3\pi}{8}\sqrt{15}\, T^{-1} \\ \frac{9\pi}{16}\sqrt{10}\, T^{-1} & \frac{112}{13} T^{-\frac{1}{2}} & -\frac{60}{77}\sqrt{10}\, T^{-\frac{1}{2}} & \frac{4}{3}\sqrt{6}\, T^{-\frac{1}{2}} \\ \frac{15}{2} T^{-1} & -\frac{60}{77}\sqrt{10}\, T^{-\frac{1}{2}} & -10\, T^{-\frac{1}{2}} & \frac{10}{7}\sqrt{15}\, T^{-\frac{1}{2}} \\ \frac{3\pi}{8}\sqrt{15}\, T^{-1} & \frac{4}{3}\sqrt{6}\, T^{-\frac{1}{2}} & \frac{10}{7}\sqrt{15}\, T^{-\frac{1}{2}} & 6\, T^{-\frac{1}{2}} \end{vmatrix} \quad .$$

With this expression it is easily shown that the asymptotic behaviour of $\mu^T(1)$ is indeed given by eq. (E.1).

Appendix F. Proof of eq. (4.34)

In this appendix we shall derive the power series expansion for μ^T given in eq. (4.34). The expansion of the first term at the right hand side of eq. (4.33) can easily be obtained from eq. (4.28) (cf. also the expression for $b_{1,1}^{(1,1)}$ for small values of T given in appendix D):

$$\hat{U} \cdot \underline{\underline{B}}^{(1,1)} \cdot \hat{U} = -\frac{2}{3}\left(1 - \frac{4}{7} T^{\frac{1}{2}} + \frac{8}{45} T^{3/2} - \frac{8}{75} T^2 + \frac{32}{1155} T^{5/2} - \frac{32}{12285} T^{7/2} \right) + O(T^4) \quad . \tag{F.1}$$

We now turn to the other term in eq. (4.33),

$$\hat{U} \cdot \underline{\underline{B}}^{(1,3)} \odot \underline{\underline{B}}^{(3,3)^{-1}} \odot \underline{\underline{B}}^{(3,1)} \cdot \hat{U} \quad .$$

Since the leading term in the expansion of $\underline{\underline{B}}^{(1,3)}$ in powers of $T^{\frac{1}{2}}$ is proportional to T (see eq. (4.18)), $\underline{\underline{B}}^{(3,3)^{-1}}$ must be expanded up to order $T^{3/2}$ and $\underline{\underline{B}}^{(1,3)}$ and $\underline{\underline{B}}^{(3,1)}$ up to order $T^{5/2}$. The tensor $\underline{\underline{B}}^{(3,1)} \cdot \hat{U}$ of rank three is traceless symmetric in its first two indices and contains only one unit vector $\hat{U}$. This tensor may therefore be decomposed in a similar way as $\underline{F}^{(3)}$ (see section 4.2). Using eqs. (4.22) - (4.24) it may be verified that

$$\underline{\underline{B}}^{(3,1)} \cdot \hat{U} = \sum_{i=1}^{3} (-1)^{i+1}\, b_{1,i}^{(1,3)}\, \underline{a}_i^{(3)} \tag{F.2}$$

and also that

$$\hat{U} \cdot \underline{\underline{B}}^{(1,3)} = \sum_{i=1}^{3} (-1)^{i+1}\, b_{1,i}^{(1,3)}\, \overleftarrow{\underline{a}_i^{(3)}} \quad . \tag{F.3}$$

Consider now the tensor $\underline{\underline{B}}^{(3,3)^{-1}}$. In view of eqs. (F.2) and (F.3) it is sufficient to determine the elements of the matrix $\underline{\beta}$ defined by the relation

$$\underline{\underline{B}}^{(3,3)^{-1}} \odot \underline{a}_i^{(3)} \equiv \sum_{j=1}^{3} (\underline{\beta})_{i,j}\, \underline{a}_j^{(3)} \quad , \quad i = 1,2,3 \tag{F.4}$$

up to order $T^{3/2}$. Eq. (F.4) is equivalent with

$$\underline{\underline{B}}^{(3,3)} \odot \underline{a}_i^{(3)} = \sum_{j=1}^{3} (\beta^{-1})_{i,j}\, \underline{a}_j^{(3)} \quad , \quad i = 1,2,3 \quad .$$

Using eq. (4.22) and the expressions for $b_{i,j}^{(3,3)}$ for small values of T, given in appendix D, one finds that

$$\beta^{-1} = \begin{pmatrix} \frac{24}{7} & \frac{32}{245}\sqrt{10}\,T & 0 \\ -\frac{32}{245}\sqrt{10}\,T & 6 & \frac{8}{105}\sqrt{15}\,T \\ 0 & -\frac{8}{105}\sqrt{15}\,T & 2 \end{pmatrix} + O(T^2) \quad .$$

Upon inversion one obtains for the matrix β :

$$\beta = \begin{pmatrix} \frac{7}{24} & -\frac{2}{315}\sqrt{10}\,T & 0 \\ \frac{2}{315}\sqrt{10}\,T & \frac{1}{6} & -\frac{2}{315}\sqrt{15}\,T \\ 0 & \frac{2}{315}\sqrt{15}\,T & \frac{1}{2} \end{pmatrix} + O(T^2) \quad . \qquad \text{(F.5)}$$

The evaluation of the term $\hat{U} \cdot \underline{\underline{B}}^{(1,3)} \odot \underline{\underline{B}}^{(3,3)^{-1}} \odot \underline{\underline{B}}^{(3,1)} \cdot \hat{U}$ is now straightforward: using eqs. (F.2) - (F.4) one obtains for this term:

$$\hat{U} \cdot \underline{\underline{B}}^{(1,3)} \odot \underline{\underline{B}}^{(3,3)^{-1}} \odot \underline{\underline{B}}^{(3,1)} \cdot \hat{U} = \sum_{i=1}^{3} \sum_{j=1}^{3} (-1)^{i+1}\, b_{1,i}^{(1,3)}\, b_{1,j}^{(1,3)}\, (\beta)_{i,j} \quad .$$

Substituting eq. (F.5) and the expressions for $b_{1,i}^{(1,3)}$ for small values of T, listed in appendix D, in this equation, one gets:

$$\hat{U} \cdot \underline{\underline{B}}^{(1,3)} \odot \underline{\underline{B}}^{(3,3)^{-1}} \odot \underline{\underline{B}}^{(3,1)} \cdot \hat{U} =$$

$$\frac{2}{3}\left(\frac{4}{225}T^2 - \frac{32}{1155}T^{5/2} - \frac{485504}{2029052025}T^3 + \frac{1216}{36855}T^{7/2} \right) + O(T^4) \quad . \qquad \text{(F.6)}$$

Substitution of eqs. (F.1) and (F.6) into eq. (4.33) yields eq. (4.34).

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Chapter IV

Drag on a sphere moving slowly in a rotating viscous fluid. Part II.

1. Introduction

In chapter 3 the drag experienced by a sphere moving slowly along the axis of a rotating incompressible fluid has been calculated in what seems to be a good approximation for all Taylor numbers and zero Reynolds number. In this chapter we shall add various, mainly analytical, details to the analysis given in chapter 3 and extend this analysis to finite Reynolds numbers by including convection in Oseen's approximation.

Several arguments can be put forward in favour of a study of the influence even of linearized convection on the values for the drag in the limit of zero Reynolds number:

1. At present it is not yet possible, even with the help of powerful computers, to solve the full Navier-Stokes equation for the problem under consideration (cf. [1]).

2. In his 1970 paper Maxworthy [2] has reported that for Rossby numbers, $Ro \equiv R/T$, less than 0.1, the drag is proportional to the product of U and Ω, as was predicted both by calculations at $R = 0$ for the inviscid fluid model [3] as well as for a viscous fluid in very rapid rotation (see e.g. ref. 4). Maxworthy noted however that the experimentally determined value for the drag was about 50% higher than predicted by these theories. But since for the values of R of Maxworthy's experiments the influence of convection was not necessarily negligible, it is not a priori clear whether this discrepancy will persist in the limit Ro tending to zero. This makes it worthwhile to carry out the present analysis.

3. Similar conclusions can also be drawn from Maxworthy's flow field measurements and visualisations: the fully linearized theory predicts a fore-aft symmetric velocity field, whereas under the experimental conditions the forward stagnant region and the rearward wake were quite different in shape. Moreover the increase in pressure on the rotation axis just in front of the sphere was less than the decrease in pressure downstream, whereas the fully linearized theory predicts that increase and decrease are of equal magnitude.

4. It has been observed by various authors (see e.g. ref. 5) that for an inviscid fluid the equations for the stream function based on the Navier-Stokes equation and the Oseen equation are identical. On the other hand viscosity plays an unimportant role at zero Reynolds number and very rapid rotation, as far as the drag is concerned [3,4]. It thus seems plausible that

for very rapid rotation and sufficiently small Reynolds numbers the values for the drag based on the Oseen equation will be close to those based on the Navier-Stokes equation.

As in chapter 3 we use for the calculation of the drag the method of induced forces. This method prescribes the introduction of an induced force density - representing the presence of the sphere - in the equation of motion for the fluid. The velocity field may then be expressed in terms of this force density, as is done in section 2 for the problem under consideration. Using this formal solution and an expansion of the induced force density in force multipoles, we derive in section 3 an expression for the mobility μ, the inverse of the friction coefficient, in the form of the quotient of two determinants. We calculate in subsection 4.1 the drag for small values of R and T, taking into account the first two force multipoles only, and compare the results with other theoretical values and experimental data. We show in subsection 4.2 that the product of the Taylor number and the mobility, based on any finite number of multipoles larger than three, may for large values of T be expanded in a power series in $T^{-\frac{1}{2}}$, with Stewartson's asymptotic result as leading term. In the same subsection we prove that for non-zero values of the Taylor number the correction term due to 'Oseen convection' is proportional to R^2, in sharp contrast with the non-rotating case, where the correction to Stokes' result is proportional to R. Finally we summarize and discuss the results derived in this chapter in section 5.

2. Formal solution for the velocity field

We consider a sphere with radius a, translationally at rest on the axis of rotation of a viscous, incompressible, unbounded fluid. The unperturbed fluid has translational velocity $\vec{U}$ in the direction of its angular velocity $\vec{\Omega}$. We assume that the motion of the fluid in the rotating frame is described adequately by the Oseen equation and the equation of continuity:

$$\rho\vec{U}\cdot\vec{\nabla}\vec{v}(\vec{r}) + 2\rho\vec{\Omega}\wedge\vec{v}(\vec{r}) + \rho\vec{\Omega}\wedge(\vec{\Omega}\wedge\vec{r}) = -\vec{\nabla}\cdot\underline{P}(\vec{r}) \qquad \text{for } r > a \tag{2.1}$$

$$\vec{\nabla}\cdot\vec{v}(\vec{r}) = 0 \tag{2.2}$$

with

$$P_{\alpha\beta} = p\,\delta_{\alpha\beta} - \eta\left(\frac{\partial v_\alpha}{\partial r_\beta} + \frac{\partial v_\beta}{\partial r_\alpha}\right) \quad . \tag{2.3}$$

The centripetal force density $\rho\vec{\Omega}\wedge(\vec{\Omega}\wedge\vec{r})$ can be incorporated into the hydrostatic pressure by defining a reduced hydrostatic pressure $p^*(\vec{r})$:

$$p^*(\vec{r}) \equiv p(\vec{r}) - \tfrac{1}{2}\rho\left(\Omega^2 r^2 - (\vec{\Omega}\cdot\vec{r})^2\right) \quad . \tag{2.4}$$

We allow the sphere to rotate freely, i.e. its angular velocity $\vec{\omega}$ is determined by the condition that the sphere experiences no torque from the fluid. In view of the symmetry of the problem $\vec{\omega}$ must be in the direction of $\vec{\Omega}$. The boundary conditions for the velocity field at the surface of the sphere and at infinity are given by:

$$\vec{v}(\vec{r}) = \vec{\omega}\wedge\vec{r} \qquad \text{for } r = a \quad , \tag{2.5}$$

$$\vec{v}(\vec{r}) = \vec{U} \qquad \text{for } r \to \infty \quad . \tag{2.6}$$

From the analysis in chapter 3 (subsection 4.1) it follows that $\vec{\omega}$ vanishes in absence of the term $\rho\vec{U}\cdot\vec{\nabla}\vec{v}(\vec{r})$ in eq. (2.1). In this case eqs. (2.1) - (2.6) are invariant under the symmetry transformations

$$\vec{r} \to -\vec{r} \quad , \quad \vec{v}(\vec{r}) \to \vec{v}(-\vec{r}) \quad , \quad p^*(\vec{r}) \to -p^*(-\vec{r}) \quad . \tag{2.7}$$

In the presence of the term $\rho\vec{U}\cdot\vec{\nabla}\vec{v}(\vec{r})$ this symmetry is broken, and $\vec{\omega}$ need not be zero. Both the imperfect fore-aft symmetry and the non-vanishing angular velocity have been observed experimentally [2,6]).

To derive the formal solution for the velocity field we use the same procedure as in section 3 of chapter 3 . We extend the validity of the equations of motion and continuity to all of space: we introduce an induced force density $\vec{F}(\vec{r})$ in the equation of motion to represent the presence of the sphere and we extend the fluid fields within the sphere. The equations of motion and continuity then read:

$$\rho\vec{U}\cdot\vec{\nabla}\vec{v}(\vec{r}) + 2\rho\vec{\Omega}\wedge\vec{v}(\vec{r}) = -\vec{\nabla}p^*(\vec{r}) + \eta\Delta\vec{v}(\vec{r}) + \vec{F}(\vec{r}) \qquad \text{for all } r \tag{2.8}$$

$$\vec{\nabla}\cdot\vec{v}(\vec{r}) = 0 \tag{2.9}$$

with $\vec{F}(\vec{r}) = 0$ for $r > a$. As extensions of the velocity field and reduced hydrostatic pressure inside the sphere we choose:

$$\vec{v}(\vec{r}) = \vec{\omega}\wedge\vec{r} \qquad \text{for } r < a \quad , \tag{2.10}$$

$$p^*(\vec{r}) = \rho(\vec{\Omega}\cdot\vec{\omega})\vec{r}\cdot(\underline{1}-\hat{\Omega}\hat{\Omega})\cdot\vec{r} \qquad \text{for } r < a \quad . \tag{2.11}$$

with $\hat{\Omega} \equiv \vec{\Omega}/\Omega$. From substitution of eqs. (2.10) and (2.11) into eq. (2.8) it follows that the induced force density must be proportional to $\delta(r-a)$, the Dirac delta function at the surface of the sphere:

$$\vec{F}(\vec{r}) = a^{-2}\,\vec{f}(\hat{r})\,\delta(r-a) \quad . \tag{2.12}$$

We now define Fourier transforms of e.g. the velocity field as

$$\vec{v}(\vec{k}) \equiv \int d\vec{r}\; e^{-i\vec{k}\cdot\vec{r}}\;\vec{v}(\vec{r}) \quad .$$

Eqs. (2.8) and (2.9) become in wavevector representation:

$$(\eta k^2 + i\rho\vec{U}\cdot\vec{k} - 2\rho\vec{\Omega}\cdot\underline{\varepsilon}\cdot)\;\vec{v}(\vec{k}) = -i\vec{k}\;p^*(\vec{k}) + \vec{F}(\vec{k}) \quad , \tag{2.13}$$

$$\vec{k}\cdot\vec{v}(\vec{k}) = 0 \quad . \tag{2.14}$$

In eq. (2.13) $\underline{\varepsilon}$ is the Levi-Civita tensor: the isotropic, completely antisymmetric, tensor of the third rank. We apply the operator $(\underline{1}-\hat{k}\hat{k})$ to both sides of eq. (2.13) and we make use of eq. (2.14). We then obtain the equation (cf. the derivation of eq. (3.13) in chapter 3)

$$\eta\left(k(k + iR\xi/a) - 2T\xi\;a^{-2}\;\hat{k}\cdot\underline{\varepsilon}\cdot\right)\vec{v}(\vec{k}) = (\underline{1}-\hat{k}\hat{k})\cdot\vec{F}(\vec{k}) \tag{2.15}$$

with $\xi \equiv \hat{U}\cdot\hat{k} \equiv \hat{\Omega}\cdot\hat{k}$*. Here the Reynolds number R and Taylor number T are defined as

* The fact that $\hat{\Omega}\cdot\hat{k}$ is strictly speaking a pseudo scalar does not influence the analysis presented in this chapter.

$$R \equiv \frac{\rho U a}{\eta} \quad , \quad T \equiv \frac{\rho \Omega a^2}{\eta} \quad .$$

The solution of eq. (2.15) is given by

$$\vec{v}(\vec{k}) = \frac{a^2}{\eta} \frac{ka(ka + iR\xi)(\underline{1}-\hat{k}\hat{k}) + 2T\xi\, \hat{k}\cdot\underline{\varepsilon}}{(ka)^2(ka + iR\xi)^2 + (2T\xi)^2} \cdot \vec{F}(\vec{k}) + (2\pi)^3\, \vec{U}\, \delta(\vec{k}) \quad . \tag{2.16}$$

This solution implies that in the rotating frame the unperturbed fluid moves with velocity $\vec{U}$.

3. Derivation of an expression for the mobility

The aim of our analysis is the evaluation of the translational mobility of the sphere. This quantity expresses the relation between the velocity $\vec{U}$ of the unperturbed fluid and the force $\vec{K}$ experienced by the sphere. In view of the symmetry of the problem under consideration the translational mobility μ is a scalar quantity, defined by the relation

$$\vec{U} \equiv \mu\, \vec{K} \quad . \tag{3.1}$$

Using eqs. (2.3), (2.4), (2.8) and (2.10) as well as Gauss' theorem, we can express $\vec{K}$ in terms of the induced force density:

$$\vec{K} = - \int_{r=a} dS\ \hat{r}\cdot\underline{P}(\vec{r}) = - \int_{r<a} d\vec{r}\ \vec{\nabla}\cdot\underline{P}(\vec{r}) = - \int d\vec{r}\ \vec{F}(\vec{r}) \quad . \tag{3.2}$$

With the help of the same equations a possible torque $\vec{T}$, exerted by the fluid on the sphere, may be related to $\vec{F}(\vec{r})$ in the following way:

$$\vec{T} = - \int_{r=a} dS\ \vec{r}\wedge\left(\hat{r}\cdot\underline{P}(\vec{r})\right) = - \int_{r<a} d\vec{r}\ \vec{r}\wedge\left(\vec{\nabla}\cdot\underline{P}(\vec{r})\right) = - \int d\vec{r}\ \vec{r}\wedge\vec{F}(\vec{r}) \quad . \tag{3.3}$$

The induced force density $\vec{F}(\vec{k})$ can be expanded in force multipoles according to (cf. eqs. (4.3) and (4.4) of chapter 3)

$$\vec{F}(\vec{k}) = \sum_{\ell=0}^{\infty} (2\ell+1)!!\ (-i)^{\ell}\ j_{\ell}(ka)\ \overline{\hat{k}^{\ell}} \odot \underline{F}^{(\ell+1)} , \tag{3.4}$$

with $\underline{F}^{(\ell+1)}$ the $(\ell+1)^{th}$ irreducible force multipole moment, given by

$$\underline{F}^{(\ell+1)} \equiv (\ell!)^{-1} \int d\hat{r} \; \overline{\hat{r}^{\ell}} \; \vec{f}(\hat{r}) \quad . \tag{3.5}$$

In order to derive a hierarchy of equations for the force multipoles we introduce the velocity surface moments, defined as

$$(2p+1)!! \; \overline{\overline{\hat{r}^{p}} \; \vec{v}(\vec{r})}^{S} \equiv (2p+1)!! \; (4\pi a^2)^{-1} \int d\vec{r} \; \overline{\hat{r}^{p}} \; \vec{v}(\vec{r}) \; \delta(r-a) \quad ,$$

$$p = 0,1,2,\dots \quad . \tag{3.6}$$

These moments become in wavevector representation (cf. chapter 3 , eq. (4.8)):

$$(2p+1)!! \; \overline{\overline{\hat{r}^{p}} \; \vec{v}(\vec{r})}^{S} = (2p+1)!! \; \frac{i^{p}}{(2\pi)^3} \int d\vec{k} \; j_p(ka) \; \overline{\hat{k}^{p}} \; \vec{v}(\vec{k}) \quad ,$$

$$p = 0,1,2,\dots \quad . \tag{3.7}$$

We now substitute expansion (3.4) into eq. (2.16) and the resulting equation into the right hand side of eq. (3.7). We evaluate the left hand side of this last equation with the boundary condition (2.5). In appendix A we show that this procedure, followed by appropriate use of the transformations $k \to -k$, $\hat{k} \to -\hat{k}$ yields the following hierarchy of equations for the force multipoles:

$$-\vec{U} \, \delta_{p,0} + a \, \underline{\varepsilon} \cdot \vec{\omega} \, \delta_{p,1} = (4\pi\eta a)^{-1} \sum_{\ell=0}^{\infty} \{ \underline{B}_1^{(p+1,\ell+1)} + \underline{B}_2^{(p+1,\ell+1)} \} \odot \underline{F}^{(\ell+1)} \quad ,$$

$$p = 0,1,2,\dots \quad , \tag{3.8}$$

with

$$\underline{B}_1^{(p+1,\ell+1)} = (2p+1)!!(2\ell+1)!! \int_{\xi>0} \frac{d\hat{k}}{2\pi} \; \overline{\hat{k}^{p}} \; (\underline{1}-\hat{k}\hat{k}) \; \overline{\hat{k}^{\ell}} \; \mathrm{Re}\, B^{(p+1,\ell+1)} \quad , \tag{3.9}$$

$$\underline{B}_2^{(p+1,\ell+1)} = (2p+1)!!(2\ell+1)!! \int_{\xi>0} \frac{d\hat{k}}{2\pi} \; \overline{\hat{k}^{p}} \; (\hat{k}\cdot\underline{\varepsilon}) \; \overline{\hat{k}^{\ell}} \; \mathrm{Im}\, B^{(p+1,\ell+1)} \tag{3.10}$$

and

$$B^{(p+1,\ell+1)} = \frac{i^{p-\ell}}{\pi} \int_{-\infty}^{\infty} d(ka) \; \frac{(ka)^2 \, j_p(ka) \, j_\ell(ka)}{(ka)^2(ka + iR\xi)^2 + 4T^2\xi^2} \left(ka(ka + iR\xi) + 2iT\xi\right)$$

$$= \frac{i^{p-\ell}}{\pi} \int_{-\infty}^{\infty} dx \frac{x^2 j_p(x)\, j_\ell(x)}{x(x+iR\xi)-2iT\xi} \quad . \tag{3.11}$$

In eqs. (3.9) and (3.10) Re and Im denote the real resp. imaginary part. In appendix A we also show how the quantity $B^{(p+1,\ell+1)}$ may be evaluated by complex integration. This evaluation yields:

$$B^{(p+1,\ell+1)} = i^{p-\ell} \left(\frac{d+ic}{c^2+d^2}\right)\left\{ \alpha^2 j_M(\alpha)\, h_m^{(1)}(\alpha) + (-1)^{p+\ell} \beta^2\, j_M(\beta)\, h_m^{(1)}(\beta) \right\} \tag{3.12}$$

with

$$\begin{aligned} \alpha &= \tfrac{1}{2} (c + id - iR\xi) \quad , \\ \beta &= \tfrac{1}{2} (c + id + iR\xi) \quad , \\ c &= \tfrac{1}{2}\sqrt{2}\, R\xi \left\{\left(1 + \frac{64\, T^2}{R^4\xi^2}\right)^{\frac{1}{2}} - 1\right\}^{\frac{1}{2}} \quad , \\ d &= \tfrac{1}{2}\sqrt{2}\, R\xi \left\{\left(1 + \frac{64\, T^2}{R^4\xi^2}\right)^{\frac{1}{2}} + 1\right\}^{\frac{1}{2}} \quad . \end{aligned} \tag{3.13}$$

In eq. (3.12) M and m denote the larger and smaller integer of the pair p and ℓ, respectively; $h_n^{(1)}(x)$ is the first spherical Bessel function of the third kind of order n (see e.g. [7])).

We shall now transform the hierarchy of tensorial equations (3.8) into an equivalent hierarchy of scalar equations. For this purpose we use the decomposition of the force multipoles given in eqs. (4.20) and (4.21) of chapter 3 (which holds both for even and odd force multipoles). Replacing therefore 2ℓ by ℓ in these equations we have (note that $\hat{\Omega} \parallel \hat{U}$ *):

$$\underline{F}^{(\ell+1)} = \sum_{j=1}^{3} F_j^{(\ell+1)}\, \underline{a}_j^{(\ell+1)} \tag{3.14}$$

with

$$\begin{aligned} \underline{a}_1^{(\ell+1)} &\equiv \left\{\frac{(2\ell+1)!!}{(\ell+1)!}\right\}^{\frac{1}{2}} \underline{\Delta}^{(\ell+1)} \odot \hat{U}^{\ell+1} \quad , \\ \underline{a}_2^{(\ell+1)} &\equiv (\ell+1)^{-\frac{1}{2}} \left\{\frac{(2\ell-1)!!}{(\ell-1)!}\right\}^{\frac{1}{2}} \underline{\square}^{(\ell)} \odot \hat{U}^{\ell} \quad , \qquad (\ell > 1) \end{aligned} \tag{3.15}$$

* We ignore the fact that $\hat{\Omega}$ is a pseudo vector (cf. the footnote on page 79).

$$\underline{a}_3^{(\ell+1)} \equiv (2\ell+1)^{-\frac{1}{2}} \left\{\frac{(2\ell-1)!!}{(\ell-1)!}\right\}^{\frac{1}{2}} \underline{\Delta}^{(\ell)} \odot \hat{U}^{\ell-1} \quad ,$$

and for $\ell = 0$

$$\underline{a}_1^{(1)} \equiv \hat{U} \quad , \quad \underline{a}_2^{(1)} \equiv 0 \quad , \quad \underline{a}_3^{(1)} \equiv 0 \quad . \tag{3.16}$$

For $\ell > 1$ the tensors $\underline{a}_j^{(\ell+1)}$ satisfy the relation (cf. chapter 3, eq. (4.22))

$$\overbrace{\underline{a}_i^{(\ell+1)} \odot} \underline{a}_j^{(\ell+1)} = (-1)^{i+1} \delta_{i,j} \quad , \quad i,j = 1,2,3 \quad . \tag{3.17}$$

From eqs. (2.12), (3.2), (3.3), (3.5), (3.14), (3.15) and (3.16) it follows that

$$\vec{K} = - \underline{F}^{(1)} = - F_1^{(1)} \hat{U} \quad , \tag{3.18}$$

$$\vec{T} = a \, \underline{\varepsilon} : \underline{F}^{(2)} = - a\sqrt{2} \, F_2^{(2)} \hat{U} \quad . \tag{3.19}$$

With the help of eqs. (3.15) and (3.16) the left hand side of eq. (3.8) may be written as

$$- \vec{U} \delta_{p,0} + a \, \vec{\omega} \cdot \underline{\varepsilon} \, \delta_{p,1} = \sum_{i=1}^{3} (- U \delta_{p,0} \delta_{i,1} + a\sqrt{2} \, \omega \, \delta_{p,1} \delta_{i,2}) \, \underline{a}_i^{(p+1)} \quad ,$$

$$p = 0,1,2,\ldots \quad . \tag{3.20}$$

We now substitute eq. (3.20) into the left hand side of eq. (3.8), eq. (3.14) into its right hand side and make use of eq. (3.17). We then obtain the following hierarchy of scalar equations:

$$4\pi\eta a \, (U \delta_{p,0} \delta_{i,1} + a\sqrt{2} \, \omega \, \delta_{p,1} \delta_{i,2}) = - \sum_{\ell=0}^{\infty} \sum_{j=1}^{3} b_{i,j}^{(p+1,\ell+1)} F_j^{(\ell+1)} \quad ,$$

$$p = 0,1,2,\ldots \quad , \quad i = 1,2,3 \tag{3.21}$$

with

$$b_{i,j}^{(p+1,\ell+1)} \equiv \overbrace{\underline{a}_i^{(p+1)} \odot} \left\{ \underline{B}_1^{(p+1,\ell+1)} + \underline{B}_2^{(p+1,\ell+1)} \right\} \odot \underline{a}_j^{(\ell+1)} \quad . \tag{3.22}$$

In appendix B we show that the quantities $b_{i,j}^{(p+1,\ell+1)}$ can be written as

$$b_{i,j}^{(p+1,\ell+1)} = \sigma\,\gamma(p,i)\,\gamma(\ell,j)\int_0^1 d\xi\,(1-\xi^2)\,P'_{2+p-i}(\xi)\,P'_{2+\ell-j}(\xi)$$

$$\times\,(\mathrm{Re},\mathrm{Im})\,B^{(p+1,\ell+1)} \quad . \qquad (3.23)$$

In eq. (3.23) σ is equal to -1 if $i=2$ and/or $j=2$, and 1 otherwise; (Re,Im) x denotes the real part of x if $i+j$ is even and the imaginary part of x if $i+j$ is odd; furthermore:

$$\gamma(p,1) = (p+1)^{-\frac{1}{2}}\,\{(2p+1)!!p!\}^{\frac{1}{2}} \qquad (p \geq 0) \quad ,$$

$$\gamma(p,2) = \{\frac{2p+1}{p+1}\}^{\frac{1}{2}}\,\{(2p+1)!!(p-1)!\}^{\frac{1}{2}} \qquad (p \geq 1) \quad , \qquad (3.24)$$

$$\gamma(p,3) = -\,\{(2p+1)!!(p-1)!\}^{\frac{1}{2}} \qquad (p \geq 1) \quad .$$

$P'_n(\xi)$ is the derivative of the Legendre polynomial of degree n. For n = 1, 2 and 3 one has explicitly

$$P'_1(\xi) = 1 \quad , \quad P'_2(\xi) = 3\xi \quad , \quad P'_3(\xi) = \frac{3}{2}\,(5\xi^2 - 1) \quad .$$

It is easily checked that the quantities $b_{i,j}^{(p+1,\ell+1)}$ satisfy the symmetry relation

$$b_{i,j}^{(p+1,\ell+1)} = (-1)^{p+\ell}\,b_{j,i}^{(\ell+1,p+1)} \quad . \qquad (3.25)$$

If we then, following a procedure also used in chapter 3, truncate the set of equations (3.21) at $p+1 = \ell+1 = N$, we can solve the remaining finite set of equations for $\vec{K} = -\,F_1^{(1)}\,\hat{U}$ by application of Cramér's rule. This yields an expression for the translational mobility which takes into account the influence of the first N force multipoles:

$$\mu(N) = (4\pi\eta a)^{-1}\begin{cases} b_{1,1}^{(1,1)} & (N = 1) \quad , \\ |b(N)|\;/\;|b'(N)| & (N \geq 2) \quad , \end{cases} \qquad (3.26)$$

where $|b(N)|$ is the determinant of the matrix $\underline{b}(N)$ with elements $b_{i,j}^{(\alpha,\beta)}$, $\alpha,\beta = 1,2,\ldots,N$, $i,j = 1,2,3$ with the proviso that $i = 1$ if $\alpha = 1,2$ and $j = 1$ if β

$= 1,2$, and $|b'(N)|$ the determinant of the matrix $\underline{b}'(N)$ with elements $b_{i,j}^{(\gamma,\delta)}$, $\gamma,\delta = 2,\ldots,N$, $i,j = 1,2,3$ with the proviso that $i = 1$ if $\gamma = 2$ and $j = 1$ if $\delta = 2$. The determinant $|b(N)|$ has the following structure:

$$\begin{vmatrix} b_{1,1}^{(1,1)} & b_{1,1}^{(1,2)} & b_{1,1}^{(1,3)} & b_{1,2}^{(1,3)} & b_{1,3}^{(1,3)} & b_{1,1}^{(1,4)} & \ldots & b_{1,3}^{(1,N)} \\ b_{1,1}^{(2,1)} & b_{1,1}^{(2,2)} & b_{1,1}^{(2,3)} & b_{1,2}^{(2,3)} & b_{1,3}^{(2,3)} & b_{1,1}^{(2,4)} & \ldots & b_{1,3}^{(2,N)} \\ b_{1,1}^{(3,1)} & b_{1,1}^{(3,2)} & b_{1,1}^{(3,3)} & b_{1,2}^{(3,3)} & b_{1,3}^{(3,3)} & b_{1,1}^{(3,4)} & \ldots & b_{1,3}^{(3,N)} \\ b_{2,1}^{(3,1)} & b_{2,1}^{(3,2)} & b_{2,1}^{(3,3)} & b_{2,2}^{(3,3)} & b_{2,3}^{(3,3)} & b_{2,1}^{(3,4)} & \ldots & b_{2,3}^{(3,N)} \\ b_{3,1}^{(3,1)} & b_{3,1}^{(3,2)} & b_{3,1}^{(3,3)} & b_{3,2}^{(3,3)} & b_{3,3}^{(3,3)} & b_{3,1}^{(3,4)} & \ldots & b_{3,3}^{(3,N)} \\ b_{1,1}^{(4,1)} & b_{1,1}^{(4,2)} & b_{1,1}^{(4,3)} & b_{1,2}^{(4,3)} & b_{1,3}^{(4,3)} & b_{1,1}^{(4,4)} & \ldots & b_{1,3}^{(4,N)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ b_{3,1}^{(N,1)} & b_{3,1}^{(N,2)} & b_{3,1}^{(N,3)} & b_{3,2}^{(N,3)} & b_{3,3}^{(N,3)} & b_{3,1}^{(N,4)} & \ldots & b_{3,3}^{(N,N)} \end{vmatrix}$$

The true mobility, μ, is obtained in the limit N tending to infinity, $\mu = \lim\limits_{N\to\infty} \mu(N)$.

4. Evaluation of the mobility

In this section we shall evaluate the translational mobility μ in two regimes: in subsection 4.1 for T and R ranging from 0 to 2.5 and in subsection 4.2 for T and R satisfying the inequality $R^2 \ll 8T$.

4.1 Small Taylor and Reynolds numbers

From the analysis of the Oseen drag using the method of induced forces for $R < 2.5$ in chapter 2 as well as from the corresponding analysis of the drag at $R = 0$ and $T < 2.5$ in chapter 3 it follows that in these limiting cases $\mu(2)$ deviates by less than one percent from the true mobility μ. It may be surmised that $\mu(2)$ is a reasonable approximation to μ over the whole regime $R < 2.5$, $T < 2.5$. This is therefore the quantity we shall evaluate in this subsection.

The explicit expression for $\mu(2)$ reads:

$$\mu(2) = (4\pi\eta a)^{-1} \left(\int_0^1 d\xi\ (1-\xi^2)\ \text{Re}\ B^{(1,1)} + \right.$$

$$\left. \left(\int_0^1 d\xi\ (1-\xi^2)\ \xi\ \text{Re}\ B^{(1,2)} \right)^2 \left(\int_0^1 d\xi\ (1-\xi^2)\ \xi^2\ \text{Re}\ B^{(2,2)} \right)^{-1} \right) \qquad (4.1)$$

with (cf. eqs. (3.12) and (3.13))

$$\text{Re}\ B^{(1,1)} = (c^2+d^2)^{-1}\{\ d + [c \sin c - d \cos c]\ e^{-d} \cosh R\xi\ \}\ ,$$

$$\text{Re}\ B^{(1,2)} = (c^2+d^2)^{-1}\{\ [d \cos c - c \sin c - \frac{c^2+d^2}{cd} \sin c]\ e^{-d} \sinh R\xi\ +$$

$$\frac{R\xi}{cd}\ (\ c - [c \cos c + d \sin c]\ e^{-d} \cosh R\xi\ )\ \}\ ,$$

$$\text{Re}\ B^{(2,2)} = (c^2+d^2)^{-1}\{\ -d\Big(1 + \frac{2}{c^2}\Big) + \Big(c \sin c - d \cos c + \frac{2(c^2+d^2)}{cd} \sin c$$

$$+ \frac{4}{c^2 d^2}\ [c^3 \sin c + d^3 \cos c]\ \Big)\ e^{-d} \cosh R\xi$$

$$+ 2\ \frac{R\xi}{cd}\ [c \cos c + d \sin c + \frac{2(c^2+d^2)}{cd} \cos c]\ e^{-d} \sinh R\xi\ \}\ .$$

The integration over ξ in eq. (4.1) has been carried out numerically. Values obtained from this expression for $D/D_S - 1$, with D/D_S the dimensionless drag defined as

$$D/D_S \equiv D/(6\pi\eta a U) = (6\pi\eta a\mu)^{-1}\ , \qquad (4.2)$$

can be found in table 1. For comparison we have also listed in this table (between brackets) values for $D/D_S - 1$, calculated with the following expression for D/D_S derived by Childress [8]):

$$D/D_S = 1 + \tfrac{3}{8}\ R \int_0^1 d\xi\ (3\xi^2 - 1)\ \{(\xi^2 + 8i\xi T/R^2)^{\frac{1}{2}} + (\xi^2 - 8i\xi T/R^2)^{\frac{1}{2}}\} + o(R)\ . \qquad (4.3)$$

In eq. (4.3) o(R) denotes a correction term which depends on R in such a way that $R^{-1}o(R)$ tends to zero in the limit R tending to zero. The above expression for D/D_S has been derived by Childress on the basis of the full Navier-Stokes equation by means of the method of matched asymptotic expansions. The validity of this expression is limited to $R \ll 1$, $T \ll 1$. From eq. (4.3) it follows that in the limits $T \to 0$ resp. $R \to 0$ the

Table 1

Values for $D/D_S - 1$, based on eqs. (4.1) and (4.3), the latter between brackets.

T \ R	0.0	0.2	0.4	0.6	0.8	1.0	1.5	2.0	2.5
0.0	0.0 (0.0)	0.07 (0.07)	0.14 (0.15)	0.21 (0.22)	0.27 (0.30)	0.33 (0.37)	0.48	0.62	0.76
0.2	0.32 (0.25)	0.32 (0.26)	0.32 (0.27)	0.33 (0.30)	0.34 (0.33)	0.35 (0.38)	0.43	0.55	0.68
0.4	0.49 (0.36)	0.49 (0.36)	0.49 (0.37)	0.49 (0.39)	0.49 (0.41)	0.49 (0.44)	0.50	0.56	0.65
0.6	0.64 (0.44)	0.64 (0.44)	0.64 (0.45)	0.63 (0.47)	0.63 (0.48)	0.62 (0.51)	0.61	0.63	0.68
0.8	0.78 (0.51)	0.78 (0.51)	0.77 (0.52)	0.77 (0.53)	0.76 (0.55)	0.75 (0.57)	0.73	0.72	0.74
1.0	0.90 (0.57)	0.90 (0.57)	0.90 (0.57)	0.89 (0.59)	0.88 (0.60)	0.88 (0.62)	0.85	0.82	0.82
1.5	1.19	1.19	1.19	1.18	1.17	1.16	1.13	1.09	1.06
2.0	1.46	1.45	1.45	1.44	1.44	1.43	1.39	1.35	1.31
2.5	1.70	1.70	1.70	1.69	1.68	1.67	1.64	1.60	1.55

dimensionless drag becomes

$$D/D_S = 1 + \frac{3}{8} R + o(R) \qquad (T = 0) \quad , \qquad (4.4)$$

$$D/D_S = 1 + \frac{4}{7} \sqrt{T} + O(T) \qquad (R = 0) \quad . \qquad (4.5)$$

Since for very small values of R the Navier-Stokes equation becomes equivalent with the Oseen equation, Childress' result for D/D_S may be compared to the results based on the present theory for $R \ll 1$.

In figure 1 we have plotted the values for $D/D_S - 1$, listed in table 1, together with Maxworthy's 1965 experimental data [6]) and the numerical results of Dennis e.a. [1]). The errors in the experimental results for $D/D_S - 1$, together with the errors due to extra- and interpolation, are estimated to add up to most about 10%. It is seen that at zero Reynolds number the values calculated with eq. (4.1) are in excellent agreement with the experimental data, which are available up to $T = 0.75$ (these results were obtained by extrapolation of data at $R < 0.2$). Childress' result however starts to deviate appreciably at $T = 0.2$.

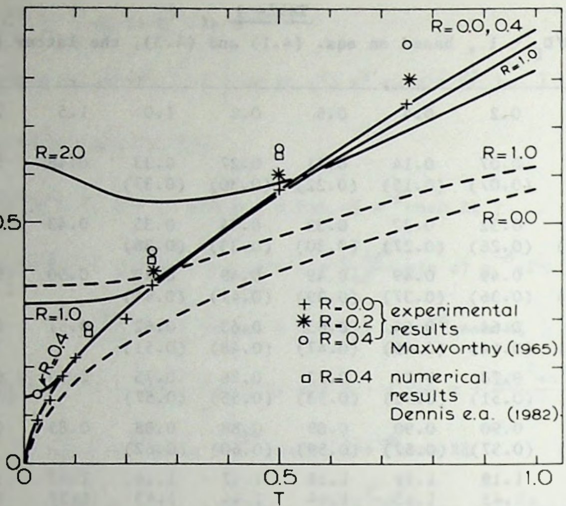

Fig. 1. $D/D_S - 1$ as function of T and R for Childress' (— — —) and our (———) result and from experiment and numerical calculation.

With respect to the numerical results of Dennis e.a. the following remarks may be made:

i.) For $T < 0.1$ and $R < 0.4$ these results agree equally well with the experimental values as the results based on eq. (4.1). For this reason the numerical results have not been plotted.

ii.) For $0.1 < T < 0.5$ the numerical results have only been plotted for $R = 0.4$. It is seen that these numerical values are in very good agreement with the experimental data. The numerical values for $D/D_S - 1$ at $R = 0$ (in fact at $R = 0.12$) have already been compared to the experimental data in section 5 of chapter 3 .

Summarizing the results of this subsection we may say that values for the drag based on $\mu(2)$ agree very well with experimental data for T up to 0.75 and $R < 0.2$, and for $T < 0.2$ and $R = 0.4$. For $R = 0.4$ and $T > 0.2$ the values calculated for $D/D_S - 1$ on the basis of $\mu(2)$ deviate by more than the estimated error from the experimental results. The possibility can not be ruled out that for these values of R and T the quantity $\mu(3)$ would yield values for the drag which agree better with the experimental data than $\mu(2)$. It seems more likely however that for these values of R and T the less satisfactory agreement is due to the fact that the Oseen equation is no longer a good approximation to the Navier-Stokes equation.

4.2 The regime $R^2 \ll 8T$

In this subsection we shall evaluate the mobility for Taylor and Reynolds numbers satisfying the condition $R^2 \ll 8T$. This includes the limit $R \to 0$ for all non-zero Taylor numbers*.

In the regime $R^2 \ll 8T$ the quantities α, β, c and d, given in eq. (3.13), may be approximated by the first few terms of a power series in $R(\xi/8T)^{\frac{1}{2}}$ $(0 < \xi < 1)$. To first order one finds

$$\begin{aligned}
\alpha &= (1+i)\sqrt{T\xi}\left(1 - \frac{1+i}{4}\frac{R}{\sqrt{T}}\sqrt{\xi} + O\left(\frac{R^2\xi}{8T}\right)\right) \quad , \\
\beta &= (1+i)\sqrt{T\xi}\left(1 + \frac{1+i}{4}\frac{R}{\sqrt{T}}\sqrt{\xi} + O\left(\frac{R^2\xi}{8T}\right)\right) \quad , \\
c &= 2\sqrt{T\xi}\left(1 + O\left(\frac{R^2\xi}{8T}\right)\right) \quad , \\
d &= 2\sqrt{T\xi}\left(1 + O\left(\frac{R^2\xi}{8T}\right)\right) \quad .
\end{aligned} \tag{4.6}$$

We now substitute the above expansions for α and β in eq. (3.12) and make use of the relation

$$f_n(z+\varepsilon) = f_n(z) + \varepsilon\left\{ f_{n-1}(z) - \frac{n+1}{z} f_n(z) \right\} + O(\varepsilon^2) \tag{4.7}$$

for spherical Bessel functions (see e.g. ref. 7). We thus obtain up to first order in $R(\xi/8T)^{\frac{1}{2}}$ the following expression for $B^{(p+1,\ell+1)}$:

* For T = 0 and R << 1 the mobility based on the Oseen equation is: $\mu = (6\pi\eta a)^{-1}\left(1 - \frac{3}{8}R + O(R^2)\right)$, as well known.

$$B^{(p+1,\ell+1)} = \begin{cases} i^{1+p-\ell}\, z\, j_M(z)\, h^{(1)}_m(z)\, \left(1 + O(\frac{R^2\xi}{8T})\right) & \text{if } p+\ell \text{ even} \;, \\ i^{p-\ell}\, \tfrac{1}{2}(1-i)\sqrt{2}\, z^2\, \{\, j_M(z)\, h^{(1)}_{m-1}(z) + j_{M-1}(z)\, h^{(1)}_m(z) & \\ \quad - \frac{p+\ell}{z}\, j_M(z)\, h^{(1)}_m(z)\, \}\, \left(\frac{R\sqrt{\xi}}{\sqrt{8T}} + O(\frac{R^2\xi}{8T})\right) & \text{if } p+\ell \text{ odd} \;, \end{cases} \tag{4.8}$$

with $z \equiv (1+i)\sqrt{T\xi}$. Using eq. (4.8) one may verify that the quantities $b^{(p+1,\ell+1)}_{i,j}$, given in eq. (3.23), can be expanded in powers of $R/\sqrt{8T}$ according to

$$b^{(p+1,\ell+1)}_{i,j} = \begin{cases} C_1(T)\left(1 + O(\frac{R^2}{8T})\right) & \text{if } p+\ell \text{ even} \;, \\ C_2(T)\left(\frac{R}{\sqrt{8T}} + O(\frac{R^2}{8T})\right) & \text{if } p+\ell \text{ odd} \;, \end{cases} \tag{4.9}$$

where $C_1(T)$ and $C_2(T)$ are functions of T only.

We now substitute eq. (4.9) into eq. (3.26). It may then easily be checked that the determinants $|b(N)|$ and $|b'(N)|$ are of the form

$$|b(N)| = g(T,N) + h(T,N)\,\frac{R^2}{8T}\left(1 + O(\frac{R}{\sqrt{8T}})\right) \;,$$

$$|b'(N)| = g'(T,N) + h'(T,N)\,\frac{R^2}{8T}\left(1 + O(\frac{R}{\sqrt{8T}})\right) \;. \tag{4.10}$$

Here $g(T,N)$, $g'(T,N)$, $h(T,N)$ and $h'(T,N)$ are functions of T and N only. Consequently, assuming that the series in $R/\sqrt{8T}$ for $\mu(N)$ converges uniformly in N, the mobility μ has for small values of $R/\sqrt{8T}$ the form

$$\mu = \lim_{N\to\infty} \mu(N) = \mu_0 + \nu(T)\,\frac{R^2}{8T} + \dots \;, \tag{4.11}$$

where μ_0 is the mobility at $R = 0$. We remark that the term linear in $R/\sqrt{8T}$ is absent in eq. (4.11). For small values of this expansion parameter Childress' result (4.3) becomes in terms of the mobility

$$\mu = (6\pi\eta a)^{-1}\,\{1 - \tfrac{4}{7}\sqrt{T}\left(1 + \frac{7}{80}\frac{R^2}{T} + O(\frac{R^4}{T^2})\right) + O(R^2)\,\} \;. \tag{4.12}$$

It is seen that expression (4.11) for μ based on the Oseen equation as well as eq. (4.12) for μ based on the Navier-Stokes equation does not contain a term proportional to R .

We note that in view of eq. (4.11) and the footnote on page 89 the derivative of μ with respect to R is not continuous at R = 0 , T = 0 :

$$\lim_{T\to 0}\lim_{R\to 0}\frac{\partial\mu}{\partial R} \neq \lim_{R\to 0}\lim_{T\to 0}\frac{\partial\mu}{\partial R} \quad .$$

This conclusion also holds for the mobility evaluated on the basis of the Navier-Stokes equation.

Consider now the mobility $\mu(N)$ for zero Reynolds number R. We shall show that this quantity, denoted by $\mu_0(N)$, may for large values of T be expanded as an asymptotic series in $T^{-\frac{1}{2}}$, and that Stewartson's result, $\mu = 3/(16\eta a T)$, is obtained as leading term in this expansion for N > 3 .

In order to obtain these results we first note that at R = 0 the quantities $B^{(p+1,\ell+1)}$ are given by (cf. eq. (4.8))

$$B^{(p+1,\ell+1)} = \begin{cases} i^{1+p-\ell}\, z\, j_M(z)\, h_m^{(1)}(z) & \text{if } p+\ell \text{ even} \quad , \qquad (4.13) \\ 0 & \text{if } p+\ell \text{ odd} \quad . \qquad (4.14) \end{cases}$$

From substitution of eq. (4.14) into eq. (3.23) it follows that the quantities $b_{i,j}^{(p+1,\ell+1)}$ vanish if $p+\ell$ odd. Hence the hierarchy of equations (3.21) can be separated into two parts (cf. chapter 3 , subsection 4.1):

$$4\pi\eta a U\,\delta_{p,0}\delta_{i,1} = -\sum_{\ell=0}^{\infty}\sum_{j=1}^{3} b_{i,j}^{(2p+1,2\ell+1)}\, F_j^{(2\ell+1)} \qquad p = 0,1,2,\ldots \;, \quad i = 1,2,3 \;, \qquad (4.15)$$

and

$$4\pi\sqrt{2}\,\eta\omega a^2\,\delta_{p,1}\delta_{i,2} = -\sum_{\ell=1}^{\infty}\sum_{j=1}^{3} b_{i,j}^{(2p,2\ell)}\, F_j^{(2\ell)} \qquad p = 1,2,3,\ldots \;, \quad i = 1,2,3 \;. \qquad (4.16)$$

As in section 3 we may solve the set of equations (4.15) for $\vec{K} = -F_1^{(1)}\hat{U}$ by truncation at $2p+1 = 2\ell+1 = 2N-1$ and application of Cramér's rule. Defining elements $c_{i,j}^{(p+1,\ell+1)} \equiv b_{i,j}^{(2p+1,2\ell+1)}$ we then obtain the following expression for the mobility based on the first N odd force multipoles (the even force multipoles do not contribute, according to eq. (4.15), at zero Reynolds number):

$$\mu_0(2N-1) = (4\pi\eta a)^{-1} \begin{cases} c^{(1,1)}_{1,1} & (N = 1) , \\ \\ |c(N)| \;/\; |c'(N)| & (N \geq 2) , \end{cases} \tag{4.17}$$

where $|c(N)|$ is the determinant of the matrix $\underline{c}(N)$ with elements $c^{(\alpha,\beta)}_{i,j}$, $\alpha,\beta = 1,2,\ldots,N$, $i,j = 1,2,3$ with the proviso that $i = 1$ if $\alpha = 1$ and $j = 1$ if $\beta = 1$, and $|c'(N)|$ the determinant of the matrix $\underline{c}'(N)$ with elements $c^{(\gamma,\delta)}_{i,j}$, $\gamma,\delta = 2,3,\ldots,N$, $i,j = 1,2,3$.

To prove that $\mu_0(2N-1)$ may for all values of N and large values of T be expanded in a power series in $T^{-\frac{1}{2}}$, we consider the quantities $B^{(2p+1,2\ell+1)}$, given in eq. (4.13). Using the relations (see e.g. ref. 7)

$$j_n(z) = (-z)^n\left(\frac{1}{z}\frac{d}{dz}\right)^n\left(\frac{e^{iz} - e^{-iz}}{2iz}\right) \quad , \quad h^{(1)}_n(z) = (-z)^n\left(\frac{1}{z}\frac{d}{dz}\right)^n \frac{e^{iz}}{iz} \quad , \tag{4.18}$$

one may verify that $B^{(2p+1,2\ell+1)}$ is of the form

$$B^{(2p+1,2\ell+1)} = (iz)^{-1}\{\, g_1\left((iz)^{-1}\right) + g_2\left((iz)^{-1}\right) e^{2iz} \,\} \tag{4.19}$$

with $g_1\left((iz)^{-1}\right)$ and $g_2\left((iz)^{-1}\right)$ polynomials in $(iz)^{-1}$ with only real coefficients. We now substitute eq. (4.19) into eq. (3.23) and replace the integration variable ξ by $y \equiv 2\sqrt{T\xi}$. This results in the following expression for the quantities $c^{(p+1,\ell+1)}_{i,j}$ (note that $z = (1+i)\sqrt{T\xi} = \frac{1}{2}(1+i)y$):

$$c^{(p+1,\ell+1)}_{i,j} = \gamma(2p,i)\,\gamma(2\ell,j)\,\frac{\sigma}{2T}\int_0^{2\sqrt{T}} y dy \left\{1-\left(\frac{y^2}{4T}\right)^2\right\} P'_{2+2p-i}\left(\frac{y^2}{4T}\right) P'_{2+2\ell-j}\left(\frac{y^2}{4T}\right)$$

$$\times\, (\mathrm{Re},\mathrm{Im})\, (iz)^{-1}\, \{\, g_1\left((iz)^{-1}\right) + g_2\left((iz)^{-1}\right) e^{-y}\, e^{iy} \,\} \quad . \tag{4.20}$$

Upon integration over y the quantities $c^{(p+1,\ell+1)}_{i,j}$ consist of polynomials in $T^{-\frac{1}{2}}$, starting with the term $T^{-\frac{1}{2}}$, and similar polynomials, multiplied by $e^{-2\sqrt{T}}\sin 2\sqrt{T}$ or $e^{-2\sqrt{T}}\cos 2\sqrt{T}$. The quantities $c^{(p+1,\ell+1)}_{i,j}$ do not contain terms of the form $T^{-n/2}\log T$, with n a positive integer, since they require the presence of terms proportional to y^{-1} in the integrand in eq. (4.20). Such terms are absent for the following reason. The terms in $(iz)^{-1}\, g_1\left((iz)^{-1}\right)$

which are proportional to y^{-4n} all have real coefficients while the terms proportional to $y^{(2-4n)}$ all have imaginary coefficients. If i+j is even, in which case the real part of $(iz)^{-1}\ g_l((iz)^{-1})$ must be chosen (cf. eq. (3.23)), the product of the two derivatives of Legendre polynomials contains only even powers of y^2, so that the integrand in eq. (4.20) does not contain the term y^{-1}. If i+j is odd the same conclusion holds: in this case the imaginary part of $(iz)^{-1}\ g_l((iz)^{-1})$ must be used and the product of derivatives of Legendre polynomials contains only odd powers of y^2. The inverse powers of y considered above do not give rise to divergences due to the lower integration boundary, since the quantities $c_{i,j}^{(p+1,\ell+1)}$ are bounded for all values of T. For large values of T this may be checked using the fact that the integrand in eq. (4.20) can be bounded by a constant for all values of y. Upon integration one obtains an upperbound for $c_{i,j}^{(p+1,\ell+1)}$ which tends to zero as $T^{-\frac{1}{2}}$ for T tending to infinity. From substitution of eq. (4.20) into eq. (4.17) it follows that the mobility consists of a power series in $T^{-\frac{1}{2}}$ and terms containing $e^{-2\sqrt{T}}$. In the limit of large values of T the latter are negligible compared to the former. This then shows that $\mu_0(2N-1)$ may indeed for all values of N and large values of T be expanded in powers of $T^{-\frac{1}{2}}$.

To prove that Stewartson's result is obtained as leading term in the asymptotic expansion of $\mu_0(2N-1)$ for $N > 2$ we apply in succession the following transformations to the quantities $c_{1,j}^{(p+1,\ell+1)}$ and $c_{i,1}^{(p+1,\ell+1)}$:

$$c_{1,j}^{(p+1,\ell+1)} \rightarrow c_{1,j}^{(p+1,\ell+1)} - \frac{\gamma(2p,1)}{\gamma(2p+2,3)}\, c_{3,j}^{(p+2,\ell+1)} \qquad \begin{matrix} p = 0,1,2,\dots,N-2\ , \\ j = 1,2,3\ , \end{matrix} \qquad (4.21)$$

$$c_{i,1}^{(p+1,\ell+1)} \rightarrow c_{i,1}^{(p+1,\ell+1)} - \frac{\gamma(2\ell,1)}{\gamma(2\ell+2,3)}\, c_{i,3}^{(p+1,\ell+2)} \qquad \begin{matrix} \ell = 0,1,2,\dots,N-2\ , \\ i = 1,2,3\ . \end{matrix} \qquad (4.22)$$

The transformation in eq. (4.21) represents the subtraction of a given multiple of the elements in the row p+2 , i=3 in the determinants $|c(N)|$ and $|c'(N)|$ in eq. (4.17) from the elements in the row p+1 , i=1 ; the transformation in eq. (4.22) represents a similar manipulation with the elements in the columns $\ell+2$, j=3 and $\ell+1$, j=1 of the determinants. The values of the new determinants $|c^*(N)|$ and $|c'^*(N)|$ are invariant under these transformations:

$$|c^*(N)| = |c(N)|\ , \qquad |c'^*(N)| = |c'(N)|\ . \qquad (4.23)$$

These determinants however now have elements $c^{*(p+1,\ell+1)}_{i,j}$ which are in terms of the old elements given by

$$c^{*(p+1,\ell+1)}_{i,j} = c^{(p+1,\ell+1)}_{i,j} - \frac{\gamma(2p,1)}{\gamma(2p+2,3)} \delta_{i,1}(1-\delta_{p,N-1})\, c^{(p+2,\ell+1)}_{3,j}$$

$$- \frac{\gamma(2\ell,1)}{\gamma(2\ell+2,3)} \delta_{j,1}(1-\delta_{\ell,N-1})\, c^{(p+1,\ell+2)}_{i,3}$$

$$+ \frac{\gamma(2p,1)\gamma(2\ell+1)}{\gamma(2p+2,3)\gamma(2\ell+2,3)} \delta_{i,1}(1-\delta_{p,N-1})\, \delta_{j,1}(1-\delta_{\ell,N-1})\, c^{(p+2,\ell+2)}_{3,3} \quad . \tag{4.24}$$

It should be remarked that the factors $1-\delta_{p,N-1}$ and $1-\delta_{\ell,N-1}$ appear only as consequence of the fact that the manipulations described above eq. (4.23) cannot be performed when $p = N-1$ and/or $\ell = N-1$. In appendix C it is shown that for $N > 2$ the quantities $c^{*(p+1,\ell+1)}_{i,j}$ behave for large values of T as

$$c^{*(p+1,\ell+1)}_{i,j} \propto \left(1-\delta_{i,1}(1-\delta_{p,N-1})\right)\left(1-\delta_{j,1}(1-\delta_{\ell,N-1})\right) T^{-\frac{1}{2}}$$

$$+ Q\left(1-\delta_{i,1}(1-\delta_{p,N-1})\, \delta_{j,1}(1-\delta_{\ell,N-1})\, (1-\delta_{p,\ell})\right) T^{-1} + O(T^{-3/2}) \quad , \tag{4.25}$$

where Q is a real number depending on p and ℓ; the quantity $c^{*(1,1)}_{1,1}$ is for $N > 2$ given by*

$$c^{*(1,1)}_{1,1} = \frac{3\pi}{4T} + O(T^{-3/2}) \quad . \tag{4.26}$$

To proceed we now introduce a second transformation:

$$c^{**(p+1,\ell+1)}_{i,j} \equiv T^{\frac{1}{2}\left(\delta_{i,1}(1-\delta_{p,N-1}) + \delta_{j,1}(1-\delta_{\ell,N-1})\right)}\, c^{*(p+1,\ell+1)}_{i,j} \quad . \tag{4.27}$$

It may be verified that the quantities $c^{**(p+1,\ell+1)}_{i,j}$ behave for large T as

* For $N = 1$ the only existing element $c^{*(1,1)}_{1,1} = c^{(1,1)}_{1,1} = b^{(1,1)}_{1,1}$ behaves asymptotically as $c^{*(1,1)}_{1,1} = 2/5\ T^{-\frac{1}{2}} + O(e^{-2\sqrt{T}})$ (cf. chapter 3, eq. (4.28)).

$$c_{i,j}^{**(p+1,\ell+1)} \propto \delta_{p,\ell}\delta_{i,1}\delta_{j,1}(1-\delta_{p,N-1}) + O(T^{-\frac{1}{2}}) \quad , \tag{4.28}$$

$$c_{1,1}^{**(1,1)} = \frac{3\pi}{4} + O(T^{-\frac{1}{2}}) \qquad (N > 2) \quad . \tag{4.29}$$

The determinants $|c^{**}(N)|$ and $|c'^{**}(N)|$ are in view of eqs. (4.23) and (4.27) given in terms of the determinants $|c(N)|$ and $|c'(N)|$ by

$$|c^{**}(N)| = T^{N-1}\,|c(N)| \quad , \quad |c'^{**}(N)| = T^{N-2}\,|c'(N)| \quad . \tag{4.30}$$

From eqs. (4.17) and (4.30) it follows that the quantity $\mu_0(2N-1)$ is for $N > 2$ given by

$$\mu_0(2N-1) = (4\pi\eta a T)^{-1}\,|c^{**}(N)| \,/\, |c'^{**}(N)| \quad . \tag{4.31}$$

Using this equation and eqs. (4.28) and (4.29) it may be checked that for large values of T one obtains the result

$$\mu_0(2N-1) = \frac{3}{16}(\eta a T)^{-1}\left(1 + \Theta(N)\,T^{-\frac{1}{2}} + O(T^{-1})\right) \quad , \tag{4.32}$$

as stated above, with $\Theta(N)$ a coefficient independent of T. This result implies that asymptotically only the monopole and quadrupole contribute.

In appendix D we have evaluated the coefficient $\Theta(N)$ for $N=2$. This evaluation yields for $\mu_0(3)$:

$$\mu_0(3) = \frac{3}{16}(\eta a T)^{-1}\left(1 + \frac{25}{69324}\left\{\frac{1429}{\pi} - \frac{6237}{4} - \frac{453357\,\pi}{256}\right\} T^{-\frac{1}{2}} + O(T^{-1})\right)$$

$$= \frac{3}{16}(\eta a T)^{-1}\left(1 - 2.4\,T^{-\frac{1}{2}} + O(T^{-1})\right) \quad . \tag{4.33}$$

We surmise that the expression for $\mu_0(3)$ given in eq. (4.33) will for large values of T be a good approximation to μ_0 for the following reason. It was shown numerically in chapter 3 that at $R = 0$ the quantity $\mu_0(3)$ differs less than 1% from $\mu_0(5)$ for <u>all</u> values of the Taylor number. This seems to indicate that the mobility converges very rapidly as a function of the number of multipoles taken into account. We further note that a rapid convergence as function of the number of multipoles taken into account has also been observed in the two related problems treated in chapters 1 and 2 .

In appendix D we have also evaluated the first term in $\mu(3)$ proportional to $R^2/8T$:

$$\mu(3) = \mu_0(3) \left(1 + \frac{R^2}{8T} \left\{\frac{60}{49\pi} T^{-\frac{1}{2}} + O(T^{-1})\right\} + O(\frac{R^3}{T^2})\right) \quad . \tag{4.34}$$

It is not known how inclusion of higher multipoles will affect the coefficient of the term proportional to R^2 in the above result.

5. Summary and discussion

The results derived in section 4 may be summarized as follows:

i.) For $R < 0.2$ and $T < 0.75$ the drag calculated on the basis of the Oseen equation is in excellent agreement with the experimental data for this quantity.

ii.) In the regime $R << \sqrt{8T}$ the Oseen equation predicts that the drag D, defined in terms of the mobility as $D \equiv \mu^{-1} U$, is given by

$$D = D_0 \left(1 + c(T) \frac{R^2}{8T} + \ldots\right) \quad , \tag{5.1}$$

with D_0 the drag at $R = 0$ and $c(T)$ a function of T only*.

iii.) It is likely that D_0 is for large values of T in good approximation given by (cf. eq. (4.33))

$$D_0 = \frac{16}{3} \eta a T U \left(1 + 2.4\ T^{-\frac{1}{2}} + O(T^{-1})\right) \quad . \tag{5.2}$$

We shall now compare the above results for the drag with experimentally obtained values for this quantity. In 1970 Maxworthy [2]) has reported that the drag behaves for small values of the Rossby number as

$$D = (8.1 \pm 0.2)\ \eta a U\ T^{(1.00 \pm 0.01)} \quad . \tag{5.3}$$

The largest Taylor number and smallest Reynolds number, on which this result is based, are $T = 445$ and $R = 2.5$.

* It may be verified, using eqs. (3.23), (4.8), and (4.18) that for large T the function $c(T)$ tends to zero (cf. eq. (4.34)).

If the Oseen equation provides a reliable description of the flow at $T = 445$ and $R = 2.5$, it may be concluded from eq. (5.1) that the drag must have been approximately equal to its value at $R = 0$. From eq. (5.2) it then follows that at $T = 445$ the quotient of drag and Taylor number would be about 12% higher than at $T = \infty$. However, from comparison of eqs. (5.2) and (5.3) it is seen that actually the value given by Maxworthy exceeds the asymptotic value at $T = \infty$ by 50%.

It is unlikely that this discrepancy is caused solely by the finite size of Maxworthy's experimental apparatus. Maxworthy himself asserts that his result has already been corrected sufficiently for end effects in the axial direction* by multiplying the drag measured at $T = 445$ and $R = 2.5$ by a factor of about 2/3. Hocking et al.[10]) have analysed this finite size effect from a theoretical point of view. They argue that Stewartson's result should at $T = 445$ be multiplied by a factor of roughly 5/3 to account for the influence of the end walls. The discrepancy would thereby be diminished from 50% to $\left(8.1 \times \frac{3}{2} / (\frac{16}{3} \times \frac{5}{3})\right) - 1 \simeq 35\%$. If we also take into account the correction of 12% for the fact that T was finite, the discrepancy is further reduced to $\left(8.1 \times \frac{3}{2} / (1.12 \times \frac{16}{3} \times \frac{5}{3})\right) - 1 \simeq 20\%$. The difference between experiment and theory is still substantial – though less than commonly assumed – and exceeds by far the experimental error of 2% claimed by Maxworthy. We may therefore not exclude the possibility that for $T = 445$ and $R = 2.5$, corresponding to the value 0.006 of the Rossby number and the value 0.04 of the expansion parameter $R/\sqrt{8T}$, the Oseen equation does not provide a satisfactory approximation to the full Navier-Stokes equation.

Appendix A. Derivation of the expressions for the connectors, eqs. (3.9) – (3.11) and (3.12)

Upon performing the substitution described in section 3 (above eq. (3.8)), the following hierarchy of equations is obtained:

* As for effects due to the presence of the cylindrical wall Maxworthy considers it unlikely that they affected the drag significantly. Recently Scott [9]) has shown that the influence of the cylindrical wall on the drag is for large T and very small R far less than 1% if the ratio of the radii of sphere and cylinder exceed 100, as was the case in Maxworthy's experiments.

$$-\vec{U}\,\delta_{p0} + a\,\underline{\varepsilon}\cdot\vec{\omega}\,\delta_{p1} = (4\pi\eta a)^{-1} \sum_{\ell=0}^{\infty} \{ \underline{B}_1^{(p+1,\ell+1)} + \underline{B}_2^{(p+1,\ell+1)} \} \odot \underline{F}^{(\ell+1)}\ , \qquad p = 0,1,2,\ldots\ , \tag{A.1}$$

with

$$\underline{B}_1^{(p+1,\ell+1)} = X \int d\vec{k}\ \frac{ka(ka + iR\xi)\ j_p(ka)\ j_\ell(ka)}{(ka)^2(ka + iR\xi)^2 + (2T\xi)^2}\ \overline{\hat{k}^p}\ (\underline{1}-\hat{k}\hat{k})\ \overline{\hat{k}^\ell}\ , \tag{A.2}$$

$$\underline{B}_2^{(p+1,\ell+1)} = X \int d\vec{k}\ \frac{2T\xi\ j_p(ka)\ j_\ell(ka)}{(ka)^2(ka + iR\xi)^2 + (2T\xi)^2}\ \overline{\hat{k}^p}\ (\hat{k}\cdot\underline{\varepsilon})\ \overline{\hat{k}^\ell} \tag{A.3}$$

and with

$$X = i^{p-\ell}\ (2p+1)!!(2\ell+1)!!\ \frac{a^3}{2\pi^2}\ .$$

The quantities $\underline{B}_1^{(p+1,\ell+1)}$ and $\underline{B}_2^{(p+1,\ell+1)}$ are invariant under the simultanously applied transformations

$$\hat{k} \to -\hat{k}\ , \qquad k \to -k\ .$$

Therefore the integration over all positive values of k may be replaced by half the integration over all real values of k. We now split up the integral over $\hat{k}$ into two parts for which $\xi \equiv \hat{U}\cdot\hat{k} > 0$ and $\xi < 0$, respectively. When we apply the above transformations to the part for which $\xi < 0$, it becomes identical with the integral for which $\xi > 0$. Hence we may replace the integration over all values of ξ between -1 and 1 by twice the integration over all values of ξ between 0 and 1 . A simple combination of the integrations over k in eqs. (A.2) and (A.3) yields eqs. (3.9) - (3.11).

We now turn to the evaluation of $B^{(p+1,\ell+1)}$, given by (see eq. (3.11)):

$$B^{(p+1,\ell+1)} = \frac{i^{p-\ell}}{\pi} \int_{-\infty}^{\infty} dx\ \frac{x^2\ j_p(x)\ j_\ell(x)}{x(x+iR\xi) - 2iT\xi} \tag{A.4}$$

If we denote the larger and smaller integer of the pair p and ℓ by M and m, respectively, and use the definitions of α and β, given in eq. (3.13), we may replace eq. (A.4) by:

$$B^{(p+1,\ell+1)} = \lim_{\varepsilon\to 0} \frac{i^{p-\ell}}{\pi} \int_{-\infty}^{\infty} dx\ \frac{x^2\ j_M(x)\ j_m(x[1+\varepsilon])}{(x-\alpha)(x+\beta)}\ . \tag{A.5}$$

The outcome of the integration does not depend on the way in which the limit is taken (i.e. $\varepsilon \uparrow 0$ or $\varepsilon \downarrow 0$), in view of the principle of dominated convergence (see e.g. ref. 11). It is easily checked that the following integrable function f(x) is a bound for the integrand:

$$f(x) = \begin{cases} \dfrac{x^2}{m'^2} & \text{for } |x| < 2M' , \\ \dfrac{4}{x^2} & \text{for } |x| > 2M' . \end{cases} \tag{A.6}$$

with m' the minimum of the pair Im α and Im β, and M' the maximum of the pair $|\alpha|$ and $|\beta|$. Using the relation

$$j_n(x) = \tfrac{1}{2} \left(h^{(1)}_n(x) + h^{(2)}_n(x) \right) ,$$

where $h^{(1)}_n(x)$ and $h^{(2)}_n(x)$ are the first and second spherical Bessel function of the third kind (see e.g. ref. 7), we obtain for $B^{(p+1,\ell+1)}$:

$$B^{(p+1,\ell+1)} = \lim_{\varepsilon \downarrow 0} \frac{i^{p-\ell}}{2\pi} \int_{-\infty}^{\infty} dx \frac{x^2 \, j_M(x) \, h^{(1)}_m(x[1+\varepsilon])}{(x-\alpha)(x+\beta)} + \lim_{\varepsilon \downarrow 0} \frac{i^{p-\ell}}{2\pi} \int_{-\infty}^{\infty} dx \frac{x^2 \, j_M(x) \, h^{(2)}_m(x[1+\varepsilon])}{(x-\alpha)(x+\beta)} . \tag{A.7}$$

To evaluate the first integral we use as contour a large semicircle above the real axis with its centre at the origin, together with that part of the real axis which joins the ends of the semicircle; for the second integral we use the reflection of this contour with respect to the real axis. The integrals round the large semicircles both tend to zero as the radii tend to infinity. Applying now Cauchy's residue theorem we obtain for the quantities $B^{(p+1,\ell+1)}$ the expression (note that Im $\alpha > 0$ and Im $\beta > 0$):

$$B^{(p+1,\ell+1)} = i^{p-\ell} \left\{ \frac{i\alpha^2}{c+id} \, j_M(\alpha) \, h^{(1)}_m(\alpha) + \frac{i\beta^2}{c+id} \, j_M(-\beta) \, h^{(2)}_m(-\beta) \right\} . \tag{A.8}$$

With the help of the relations (see e.g. ref. 7)

$$j_n(-x) = (-1)^n \, j_n(x) \quad , \quad h^{(2)}_n(-x) = (-1)^n \, h^{(1)}_n(x) \quad ,$$

it is easily shown that eq. (A.8) is equivalent with eq. (3.12).

Appendix B. Derivation of eq. (3.23)

For the derivation of eq. (3.23) we shall make use of the relations

$$\underline{\Delta}^{(\ell)} \odot^{*} \underline{\Delta}^{(\ell)} = \frac{2\ell+1}{2\ell-1}\, \underline{\Delta}^{(\ell-1)} \quad , \tag{B.1}$$

$$\underline{\square}^{(\ell)} \odot^{*} \underline{\square}^{(\ell)} = -\frac{\ell+1}{\ell}\, \underline{\Delta}^{(\ell)} \quad , \tag{B.2}$$

$$\overline{\hat{k}^{\ell}} \odot \overline{\hat{U}^{\ell+1}} = \frac{\ell!}{(2\ell+1)!!} \left\{ \hat{U}\, P'_{\ell+1}(\hat{k}\cdot\hat{U}) - \hat{k}\, P'_{\ell}(\hat{k}\cdot\hat{U}) \right\} \quad , \tag{B.3}$$

$$\overline{\hat{k}^{\ell}} \odot \underline{\square}^{(\ell)} \odot \overline{\hat{U}^{\ell}} = \frac{(\ell-1)!}{(2\ell-1)!!}\, \hat{k}\cdot\underline{\varepsilon}\cdot\hat{U}\, P'_{\ell}(\hat{k}\cdot\hat{U}) \quad . \tag{B.4}$$

In eqs. (B.1) and (B.2), which are identical with eqs. (2.15) and (2.31) of ref. 12, $\odot^{*}$ denotes the $\ell + 1$ fold contraction. Using eqs. (1.7) and (2.55) of ref. 12 , viz.:

$$\overline{\hat{k}^{\ell}} \odot \overline{\hat{U}^{\ell}} = \frac{\ell!}{(2\ell-1)!!}\, P_{\ell}(\hat{k}\cdot\hat{U}) \tag{B.5}$$

and

$$\overline{\hat{U}^{\ell+1}} = \overline{\hat{U}^{\ell}}\, \hat{U} - \frac{\ell}{2\ell+1}\, \underline{\Delta}^{(\ell)} \odot \overline{\hat{U}^{\ell-1}} \quad , \tag{B.6}$$

and the summation theorem for Legendre polynomials (see e.g. ref. 13 , form. (8.915.2))

$$P'_{\ell}(\xi) = \sum_{n=0} (2\ell - 4n - 1)\, P_{\ell-2n-1}(\xi) \quad , \tag{B.7}$$

where the summation is cut off at the first term with negative subscript, eq. (B.3) may be derived in the following way:

$$\overline{\hat{k}^{\ell}} \odot \overline{\hat{U}^{\ell+1}} = \overline{\hat{k}^{\ell}} \odot \overline{\hat{U}^{\ell}}\, U - \frac{\ell}{2\ell+1}\, \overline{\hat{k}^{\ell}} \odot \overline{\hat{U}^{\ell-1}}$$

$$= \frac{\ell!}{(2\ell-1)!!}\, \hat{U}\, P_{\ell}(\hat{k}\cdot\hat{U}) - \frac{\ell}{2\ell+1}\, \hat{k}\, \overline{\hat{k}^{\ell-1}} \odot \overline{\hat{U}^{\ell-1}} + \frac{\ell(\ell-1)}{(2\ell+1)(2\ell-1)}\, \overline{\hat{k}^{\ell-2}} \odot \overline{\hat{U}^{\ell-1}}$$

$$= \frac{\ell!}{(2\ell+1)!!} \left\{ (2\ell+1)\, \hat{U}\, P_{\ell}(\hat{k}\cdot\hat{U}) - (2\ell-1)\, \hat{k}\, P_{\ell-1}(\hat{k}\cdot\hat{U}) \right\}$$

$$+ \frac{\ell(\ell-1)}{(2\ell+1)(2\ell-1)} \overline{\hat{k}^{\ell-2}} \odot \overline{\hat{U}^{\ell-1}}$$

$$= \frac{\ell!}{(2\ell+1)!!} \sum_{n=0} \left\{ (2\ell+1-4n)\, \hat{U}\, P_{\ell-2n}(\hat{k}\cdot\hat{U}) - (2\ell-1-4n)\, \hat{k}\, P_{\ell-1-2n}(\hat{k}\cdot\hat{U}) \right\}$$

$$= \frac{\ell!}{(2\ell+1)!!} \left\{ \hat{U}\, P'_{\ell+1}(\hat{k}\cdot\hat{U}) - \hat{k}\, P'_{\ell}(\hat{k}\cdot\hat{U}) \right\} .$$

With the help of eqs. (B.3), (B.5) – (B.7), the definition of $\underline{\square}^{(\ell)}$ given in chapter 3, subsection 4.2 and eq. (2.27) of ref. 12 , viz.:

$$(\underline{\square}^{(\ell)})_{\mu_1,..,\mu_\ell,\lambda,\nu_1,..,\nu_\ell} = \frac{(\ell+1)(2\ell+1)}{\ell(2\ell+3)} (\underline{\Delta}^{(\ell+1)})_{\mu_1,..,\mu_\ell,\mu,\nu,\nu_1,..,\nu_\ell} (\underline{\varepsilon})_{\mu,\nu,\lambda} \quad , \tag{B.8}$$

one may derive eq. (B.4) as follows ($\bullet$ denotes the ℓ-1 fold contraction):

$$\overline{\hat{k}^{\ell}} \odot \underline{\square}^{(\ell)} \odot \overline{\hat{U}^{\ell}} = \underline{\varepsilon}:(\overline{\hat{k}^{\ell}} \bullet \overline{\hat{U}^{\ell}})$$

$$= \underline{\varepsilon}:\Big(\hat{k}\hat{U}\, \overline{\hat{k}^{\ell-1}} \bullet \overline{\hat{U}^{\ell-1}} - \frac{\ell-1}{2\ell-1} \left[\overline{\hat{k}^{\ell-2}} \odot \overline{\hat{U}^{\ell-1}}\, \hat{U} + \hat{k}\, \overline{\hat{k}^{\ell-1}} \odot \overline{\hat{U}^{\ell-2}} \right]$$

$$+ \left(\frac{\ell-1}{2\ell-1}\right)^2 \overline{\hat{k}^{\ell-2}} \odot \underline{\Delta}^{(\ell-1)} \odot \overline{\hat{U}^{\ell-2}} \Big)$$

$$= \frac{(\ell-1)!}{(2\ell-1)!!} \hat{k}\cdot\underline{\varepsilon}\cdot\hat{U} \left((2\ell-1)\, P_{\ell-1}(\hat{k}\cdot\hat{U}) + 2\, P'_{\ell-2}(\hat{k}\cdot\hat{U}) \right)$$

$$- \frac{(\ell-1)(\ell-2)}{(2\ell-1)(2\ell-3)} \overline{\hat{k}^{\ell-2}} \odot \underline{\square}^{(\ell-2)} \odot \overline{\hat{U}^{\ell-2}}$$

$$= \frac{(\ell-1)!}{(2\ell-1)!!} \hat{k}\cdot\underline{\varepsilon}\cdot\hat{U}\, P'_{\ell}(\hat{k}\cdot\hat{U}) + \frac{(\ell-1)(\ell-2)}{(2\ell-1)(2\ell-3)} \times$$

$$\left(\frac{(\ell-3)!}{(2\ell-5)!!} \hat{k}\cdot\underline{\varepsilon}\cdot\hat{U}\, P'_{\ell-2}(\hat{k}\cdot\hat{U}) - \overline{\hat{k}^{\ell-2}} \odot \underline{\square}^{(\ell-2)} \odot \overline{\hat{U}^{\ell-2}} \right)$$

$$= \frac{(\ell-1)!}{(2\ell-1)!!} \hat{k}\cdot\underline{\varepsilon}\cdot\hat{U}\, P'_{\ell}(\hat{k}\cdot\hat{U}) \quad .$$

Using eqs. (3.15) and (B.1) – (B.4) one may verify that the following identities hold for $\ell > 1$:

$$\overbrace{\hat{k}^{\ell}} \odot \underline{a}_1^{(\ell+1)} = (\ell+1)^{-\frac{1}{2}} \left\{\frac{\ell!}{(2\ell+1)!!}\right\}^{\frac{1}{2}} \left(\hat{U}\, P'_{\ell+1}(\hat{k}\cdot\hat{U}) - \hat{k}\, P'_{\ell}(\hat{k}\cdot\hat{U}) \right) \quad , \tag{B.9}$$

$$\overbrace{\hat{k}^{\ell}} \odot \underline{a}_2^{(\ell+1)} = (\ell+1)^{-\frac{1}{2}} \left\{\frac{(\ell-1)!}{(2\ell-1)!!}\right\}^{\frac{1}{2}} \hat{k}\cdot\underline{\varepsilon}\cdot\hat{U}\, P'_{\ell}(\hat{k}\cdot\hat{U}) \quad , \tag{B.10}$$

$$\overbrace{\hat{k}^{\ell}} \odot \underline{a}_3^{(\ell+1)} = \left\{\frac{(\ell-1)!}{(2\ell+1)!!}\right\}^{\frac{1}{2}} \left(\hat{k}\, P'_{\ell}(\hat{k}\cdot\hat{U}) - \hat{U}\, P'_{\ell-1}(\hat{k}\cdot\hat{U}) \right) \quad . \tag{B.11}$$

With the help of eqs. (3.9), (3.10), (3.17) and (B.9) - (B.11) expression (3.23) for the quantities $b_{i,j}^{(p+1,\ell+1)}$ may now be derived from eq. (3.22) in the following way:

$$b_{i,j}^{(p+1,\ell+1)} \equiv \underline{a}_i^{(p+1)} \odot \left\{ \underline{B}_1^{(p+1,\ell+1)} + \underline{B}_2^{(p+1,\ell+1)} \right\} \odot \underline{a}_j^{(\ell+1)}$$

$$= (2p+1)!!(2\ell+1)!! \int_{\xi>0} \frac{d\hat{k}}{2\pi}$$

$$\underline{a}_i^{(p+1)} \odot \overbrace{\hat{k}^{p}} \left\langle \begin{matrix} \underline{1}-\hat{k}\hat{k} \\ \hat{k}\cdot\underline{\varepsilon} \end{matrix} \right\rangle \overbrace{\hat{k}^{\ell}} \odot \underline{a}_j^{(\ell+1)} \left\langle \begin{matrix} \mathrm{Re}\ B^{(p+1,\ell+1)} \\ \mathrm{Im}\ B^{(p+1,\ell+1)} \end{matrix} \right.$$

$$= \sigma\, \gamma(p,i)\, \gamma(\ell,j) \int_0^1 d\xi\, (1-\xi^2)\, P'_{2+p-i}(\xi)\, P'_{2+\ell-j}(\xi)$$

$$\times\ (\mathrm{Re},\mathrm{Im})\ B^{(p+1,\ell+1)} \quad .$$

Appendix C. Derivation of eqs. (4.25) and (4.26)

In this appendix we shall show that the transforms $c_{i,j}^{*(p+1,\ell+1)}$, given in eq. (4.24), behave for large values of T according to

$$c_{i,j}^{*(p+1,\ell+1)} \propto (1-\delta_{i,1})(1-\delta_{j,1})\, T^{-\frac{1}{2}} + Q\big(1-\delta_{i,1}\delta_{j,1}(1-\delta_{p,\ell})\big)\, T^{-1}$$

$$+\ O(T^{-3/2}) \quad , \quad p,\ell = 0,1,2,\ldots \quad , \tag{C.1}$$

$$c_{1,1}^{*(1,1)} = \frac{3\pi}{4T} + O(T^{-3/2}) \quad , \tag{C.2}$$

where Q denotes a real number depending on p and ℓ. Eqs. (C.1) and (C.2) become identical with eqs. (4.25) and (4.26) upon replacing $\delta_{i,1}$ by $\delta_{i,1}(1-\delta_{p,N-1})$ and $\delta_{j,1}$ by $\delta_{j,1}(1-\delta_{\ell,N-1})$.

Using the recurrence relation for spherical Bessel functions (ref. 7 , eq. (10.1.19)),

$$f_n(z) + f_{n+2}(z) = \frac{2n+3}{z} f_{n+1}(z) \quad , \quad n = 0,\pm1,\pm2,\ldots \quad ,$$

and a formula for the cross product of spherical Bessel functions, viz.:

$$j_n(z)\, h^{(1)}_{n-1}(z) - j_{n-1}(z)\, h^{(1)}_n(z) = \frac{i}{z^2} \ ,$$

(which is a modified version of eq. (10.1.31) of ref. 7), the transforms $c^{*(p+1,\ell+1)}_{i,j}$ may be written as

$$c^{*(p+1,\ell+1)}_{i,j} = \sigma\, \gamma(2p,i)\, \gamma(2\ell,j)\, \frac{1}{2T} \int_0^{2\sqrt{T}} y\,dy\, \left(1-\left(\frac{y^2}{4T}\right)^2\right)$$
$$\times\, P'_{2+2p-i}\left(\frac{y^2}{4T}\right) P'_{2+2\ell-j}\left(\frac{y^2}{4T}\right) (\mathrm{Re},\mathrm{Im})\ B^{*(2p+1,2\ell+1)} \qquad (C.3)$$

with

$$B^{*(2p+1,2\ell+1)} = (-1)^{(p-\ell)}\, (4p+3)^{\delta_{i,1}} (4\ell+3)^{\delta_{j,1}}$$
$$\times\, z^{-\delta_{i,1}-\delta_{j,1}} \left(iz\, j_{M'}(z)\, h^{(1)}_{m'}(z) - \frac{\delta_{p,\ell}\delta_{i,1}\delta_{j,1}}{4p+3} \right) \quad . \qquad (C.4)$$

Here $z = \frac{1}{2}(1+i)y$ while M' and m' denote the larger and smaller integer of the pair $2p+\delta_{i,1}$ and $2\ell+\delta_{j,1}$, respectively. With the help of the relations (4.18) for $j_n(z)$ and $h^{(1)}_n(z)$ one may verify that the quantities $B^{*(2p+1,2\ell+1)}$ are of the form (cf. eq. (4.19))

$$B^{*(2p+1,2\ell+1)} = z^{-(1+\delta_{i,1}+\delta_{j,1})}\{h_1(z^{-1}) + h_2(z^{-1})\, e^{2iz} + z\delta_{i,1}\delta_{j,1}\} \qquad (C.5)$$

where $h_1(z^{-1})$ and $h_2(z^{-1})$ are power series in z^{-1}. It follows from substitution of eq. (C.5) into eq. (C.3) that the quantities $c^{*(p+1,\ell+1)}_{i,j}$ behave asymptotically as

$$c^{*(p+1,\ell+1)}_{i,j} \propto (1-\delta_{i,1})(1-\delta_{j,1})\, T^{-\frac{1}{2}} + O(T^{-1}) \quad . \qquad (C.6)$$

To prove the validity of eq. (C.1) it is now sufficient to show that

$$\lim_{T\to\infty} T\, c^{*(p+1,\ell+1)}_{1,1} = Q\, \delta_{p,\ell} \quad . \tag{C.7}$$

According to eqs. (C.3) and (C.4) the quantity $c^{*(p+1,\ell+1)}_{1,1}$ is given by

$$c^{*(p+1,\ell+1)}_{1,1} = (-1)^{(p-\ell)} (4p+3)(4\ell+3)\, \gamma(2p,1)\gamma(2\ell,1)\, \frac{1}{T}\, \mathrm{Re} \int_0^{2\sqrt{T}} \frac{dy}{y}$$

$$\left\{1-\left(\frac{y^2}{4T}\right)^2\right\} P'_{2p+1}\left(\frac{y^2}{4T}\right) P'_{2\ell+1}\left(\frac{y^2}{4T}\right) \left(z\, j_{M'}(z)\, h^{(1)}_{m'}(z) + \frac{i\delta_{p,\ell}}{4p+3} \right) \quad . \tag{C.8}$$

We note that the term proportional to $i\delta_{p,\ell}$ does not contribute. In order to show that eq. (C.7) holds we shall now consider in detail the integral

$$I = \mathrm{Re} \int_0^{2\sqrt{T}} \frac{dy}{y} \left\{1-\left(\frac{y^2}{4T}\right)^2\right\} P'_{2p+1}\left(\frac{y^2}{4T}\right) P'_{2\ell+1}\left(\frac{y^2}{4T}\right) z\, j_{M'}(z)\, h^{(1)}_{m'}(z)$$

in the limit T tending to infinity. Replacing the integration variable y by z = ½(1+i)y, this integral is given by

$$I = \mathrm{Re} \int_0^{(1+i)\sqrt{T}} dz \left\{1+\left(\frac{z^2}{2T}\right)^2\right\} P'_{2p+1}\left(-\frac{iz^2}{2T}\right) P'_{2\ell+1}\left(-\frac{iz^2}{2T}\right) j_{M'}(z)\, h^{(1)}_{m'}(z)$$

$$= \lim_{\varepsilon\to 0} \mathrm{Re} \int_0^{(1+i)\sqrt{T}} dz \left\{1+\left(\frac{z^2}{2T}\right)^2\right\} P'_{2p+1}\left(-\frac{iz^2}{2T}\right) P'_{2\ell+1}\left(-\frac{iz^2}{2T}\right)$$

$$\times\, j_{M'}(z)\, h^{(1)}_{m'}(z[1+\varepsilon]) \tag{C.9}$$

where the integration path is the line segment between the points (0,0) and $(\sqrt{T},\sqrt{T})$ in the complex plane. Since $P'_n(x)$ is for all positive integer values of n bounded for $0 < x < 1$, the value of I does not depend on the way in which the limit $\varepsilon \to 0$ is taken, i.e. $\varepsilon \uparrow 0$ or $\varepsilon \downarrow 0$, in view of the principle of dominated convergence (cf. appendix A); an integrable function which bounds the integrand is easily constructed. To evaluate I we first use the contour consisting of the segments

$$\left.\begin{aligned} z &= x \\ z &= (1+i)\tfrac{1}{2}\sqrt{2}\, x \end{aligned}\right\} \; 0 < x < \sqrt{2T} \quad ,$$

$$z = \sqrt{2T}\, e^{i\phi} \qquad 0 < \phi < \frac{\pi}{4} \quad . \tag{C.10}$$

The integrand in eq. (C.9) has at most a simple pole at $z = 0$, since for small values of z $j_n(z)$ and $h_n^{(1)}(z)$ are given by

$$j_n(z) = \frac{z^n}{(2n+1)!!} + O(z^{n+2}) \quad , \quad h_n^{(1)}(z) = -i \frac{(2n-1)!!}{z^{n+1}} + O(z^{-n}) \quad . \tag{C.11}$$

Using the fact that in the limit $T \to \infty$ the integral over the circle segment in (C.10) vanishes, the integral I may be written in the form

$$I = \lim_{\varepsilon \downarrow 0} \mathrm{Re} \left(-\frac{\pi i}{4} (\text{residue in } z=0) + \int_0^{\sqrt{2T}} dz \right)\left\{1+\left(\frac{z^2}{2T}\right)^2\right\}$$

$$\times P'_{2p+1}\left(-\frac{iz^2}{2T}\right) P'_{2\ell+1}\left(-\frac{iz^2}{2T}\right) j_{M'}(z)\, h^{(1)}_{m'}(z[1+\varepsilon]) \quad . \tag{C.12}$$

Since $M'+m'$ is even, the integrand in eq. (C.12) is an even function of z and therefore the integral from 0 to $\sqrt{2T}$ may be replaced by half the integral from $-\sqrt{2T}$ to $\sqrt{2T}$; this last integral can be evaluated using as contour the large semicircle above the real axis with its centre at the origin and with radius $\sqrt{2T}$, together with the part of the real axis between $-\sqrt{2T}$ and $\sqrt{2T}$. In the limit $T \to \infty$ the integral round the large semicircle vanishes. We now apply Cauchy's residue theorem and make use of eq. (C.11). We then obtain for I:

$$I = \mathrm{Re}\, \frac{\pi i}{4} (\text{residue in } z=0) \left\{1+\left(\frac{z^2}{2T}\right)^2\right\} P'_{2p+1}\left(-\frac{iz^2}{2T}\right) P'_{2\ell+1}\left(-\frac{iz^2}{2T}\right)$$

$$\times j_{M'}(z)\, h^{(1)}_{m'}(z)$$

$$= \delta_{p,\ell} \left(P'_{2p+1}(0)\right)^2 \frac{\pi}{4(4p+3)}$$

$$= \delta_{p,\ell} \left(\frac{(2p+1)!!}{(2p)!!}\right)^2 \frac{\pi}{4(4p+3)} \quad . \tag{C.13}$$

In the last member of eq. (C.13) we have used formula (8.911.3) of ref. 13, viz.:

$$P_{2p+1}(z) = (-1)^p \frac{(2p+1)!!}{(2p)!!} z + O(z^3) \quad .$$

From eqs. (C.8), (C.9) and (C.13) it follows that

$$c^{*(p+1,\ell+1)}_{1,1} \propto \delta_{p,\ell}\, T^{-1} \frac{\pi}{4} \frac{(4p+3)!!}{2p+1} \left(\frac{(2p+1)!!}{(2p)!!}\right)^2 (2p)! + O(T^{-3/2}) \quad . \tag{C.14}$$

Combination of eqs. (C.6) and (C.14) yields eqs. (C.1) and (C.2).

Appendix D. Derivation of eqs. (4.33) and (4.34)

The derivation of eqs. (4.33) and (4.34) is a matter of straightforward calculation of integrals.

In view of the symmetry property (3.25) we have to evaluate the quantities $b_{i,j}^{(p+1,\ell+1)}$ for large Taylor numbers up to first order in R, for p,ℓ = 0,1,2 with $p < \ell$ and, if $p = \ell$, only for $i < j$. Leaving terms proportional to $\exp(-2\sqrt{T})$ out of consideration the required quantities are:

$$b_{1,1}^{(1,1)} = \frac{2}{5} T^{-\frac{1}{2}} ,$$

$$b_{1,1}^{(1,2)} = \frac{1}{28} \sqrt{6}\, R\, T^{-3/2} ,$$

$$b_{1,1}^{(1,3)} = -\frac{4}{15} \sqrt{10}\, T^{-\frac{1}{2}} + \frac{9\pi}{16} \sqrt{10}\, T^{-1} - \frac{18}{7} \sqrt{10}\, T^{-3/2} ,$$

$$b_{1,2}^{(1,3)} = -\frac{10}{7} T^{-\frac{1}{2}} + \frac{15}{2} T^{-1} - 9\, T^{-3/2} ,$$

$$b_{1,3}^{(1,3)} = -\frac{2}{5} \sqrt{15}\, T^{-\frac{1}{2}} + \frac{3\pi}{8} \sqrt{15}\, T^{-1} - \sqrt{15}\, T^{-3/2} ,$$

$$b_{1,1}^{(2,2)} = \frac{3}{5} T^{-\frac{1}{2}} + \frac{9}{14} T^{-3/2} ,$$

$$b_{1,1}^{(2,3)} = \frac{6}{77} \sqrt{15}\, R\, T^{-3/2} + \frac{9}{10} \sqrt{15}\, R\, T^{-5/2} ,$$

$$b_{1,2}^{(2,3)} = \frac{1}{2} \sqrt{6}\, R\, T^{-3/2} + \frac{135}{56} \sqrt{6}\, R\, T^{-5/2} ,$$

$$b_{1,3}^{(2,3)} = -\frac{3}{14} \sqrt{10}\, R\, T^{-3/2} + \frac{81}{40} \sqrt{10}\, R\, T^{-5/2} ,$$

$$b_{1,1}^{(3,3)} = \frac{112}{13} T^{-\frac{1}{2}} - \frac{360}{11} T^{-3/2} + 180\, T^{-5/2} ,$$

$$b_{1,2}^{(3,3)} = -\frac{60}{77} \sqrt{10}\, T^{-\frac{1}{2}} - 6 \sqrt{10}\, T^{-3/2} + \frac{405}{7} \sqrt{10}\, T^{-5/2} ,$$

$$b_{1,3}^{(3,3)} = \frac{4}{3} \sqrt{6}\, T^{-\frac{1}{2}} - \frac{90}{7} \sqrt{6}\, T^{-3/2} + 45 \sqrt{6}\, T^{-5/2} ,$$

$$b_{2,2}^{(3,3)} = -10\, T^{-\frac{1}{2}} - \frac{225}{7} T^{-3/2} + \frac{405}{2} T^{-5/2} ,$$

$$b_{2,3}^{(3,3)} = \frac{10}{7} \sqrt{15}\, T^{-\frac{1}{2}} - 9 \sqrt{15}\, T^{-3/2} + \frac{45}{2} \sqrt{15}\, T^{-5/2} ,$$

$$b^{(3,3)}_{3,3} = 6\,T^{-\frac{1}{2}} - \frac{15}{2}\,T^{-3/2} + \frac{75}{4}\,T^{-5/2} \quad .$$

We now substitute the above results into eq. (3.26) and, using the fact that $c^{(p+1,\ell+1)}_{i,j} = b^{(2p+1,2\ell+1)}_{i,j}$, also into eq. (4.17). Upon evaluation of the determinants one obtains the results given in eqs. (4.33) and (4.34).

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SAMENVATTING

Het doel van het onderzoek beschreven in dit proefschrift is de bestudering van de weerstand die een bol ondervindt bij langzame beweging langs de as van een roterende vloeistof. In het bijzonder wordt onderzocht hoe deze weerstand afhangt van het Taylor getal, de dimensieloze maat voor de rotatiesnelheid van de vloeistof.

De meest gangbare methode in de hydrodynamica om de weerstand te berekenen, die een object ondervindt bij beweging door een vloeistof, vereist het expliciet oplossen van de vloeistofvelden - het snelheidsveld en het drukveld - uit de bewegingsvergelijking voor de vloeistof. Uit zowel experimentele waarnemingen als theoretische beschouwingen blijkt dat de vloeistofvelden een nogal gecompliceerde structuur hebben in het in dit proefschrift bestudeerde probleem. Het is derhalve niet verrassend dat men slechts in die gevallen, waarin de structuur van de vloeistofvelden relatief eenvoudig is, nl. bij zeer kleine en zeer grote waarden van het Taylor getal, de weerstand heeft kunnen berekenen met behulp van bovengenoemde methode.

In deze dissertatie wordt met een alternatieve methode de weerstand die de bol ondervindt systematisch berekend in successieve benaderingen voor alle waarden van het Taylor getal. Deze methode, die het mogelijk maakt om de weerstand te berekenen zonder kennis van de expliciete oplossing van de bewegingsvergelijking voor de vloeistof, is gebaseerd op de door P. Mazur en W. van Saarloos gegeven methode om de hydrodynamische interacties tussen een willekeurig aantal bollen te analyseren.

De methode berust op de invoering van een geïnduceerde krachtdichtheid in de bewegingsvergelijking van de vloeistof. De vloeistofvelden kunnen dan formeel worden opgelost in termen van deze krachtdichtheid. Door nu de geïnduceerde krachtdichtheid te ontwikkelen in krachtmultipolen en op geschikte wijze gebruik te maken van de randvoorwaarden aan het oppervlak van de bol kan men een hierarchie van vergelijkingen opstellen voor deze multipolen. Hieruit kan door eliminatie de weerstandskracht (de eerste krachtmultipool) worden opgelost.

Om de bruikbaarheid van de methode aan te tonen wordt deze allereerst in hoofdstukken I en II toegepast op twee klassieke hydrodynamische problemen. In hoofdstuk I berekenen we de kracht op een oneindig lange cilinder, in rust in een uniforme, stationaire stroming loodrecht op de as van de cilinder. In hoofdstuk II wordt de kracht uitgerekend op een bol, in rust in eenzelfde stroming. In hoofdstuk III wordt vervolgens de weerstand berekend die een bol ondervindt bij langzame beweging langs de as van een roterende vloeistof. Tenslotte wordt in hoofdstuk IV onderzocht hoe de in hoofdstuk III berekende weerstand verandert bij toenemende translatiesnelheid van de bol.

CURRICULUM VITAE

van Anton Jacobus Weisenborn
geboren op 3 juni 1959 te Voorburg

Na het behalen van het einddiploma Gymnasium β aan het Sint Maartens College te Voorburg ben ik in 1977 begonnen aan mijn studie aan de Rijksuniversiteit te Leiden. In februari 1980 legde ik het kandidaatsexamen Natuurkunde en Wiskunde met Scheikunde af, in juni 1982 het doctoraalexamen Natuurkunde met bijvak Wiskunde en onderwijsbevoegdheid in de natuurkunde. Tijdens de studie voor het doctoraalexamen deed ik experimenteel werk in de groep van Prof. dr. J.J.M. Beenakker. Als opdracht voor het doctoraal examen verrichtte ik onder leiding van Prof. dr. P. Mazur onderzoek naar de weerstand, die een bol c.q. cilinder ondervindt bij beweging door een vloeistof.

Vanaf september 1982 ben ik als wetenschappelijk assistent van Prof. dr. P. Mazur werkzaam geweest aan het Instituut-Lorentz. Onder diens leiding heb ik het bovengenoemde onderzoek voortgezet en uitgebreid. Een groot deel van de resultaten van dit onderzoek zijn weergegeven in dit proefschrift.

Aan het onderwijs droeg ik onder meer bij door het geven van werkcolleges bij het college Statistische Fysica van Prof. dr. D. Bedeaux. Daarnaast verleende ik assistentie bij het vervaardigen van het collegedictaat behorende bij dit college en bij het afnemen van verscheidene tentamens.

Tijdens het onderzoek werd ik in staat gesteld om de volgende conferenties te bezoeken:

- 16^{th} Symposium on Advanced Problems and Methods in Fluid Mechanics, Spala, Polen, 4 - 10 september 1983,
- 16^{th} International Congres on Theoretical and Applied Mechanics, Lyngby, Denemarken, 19 - 25 augustus 1984.

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A.J. Weisenborn,
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Journal of Fluid Mechanics 153 (1985) 215 .

STELLINGEN

(1) De kracht die een bol met straal a ondervindt bij beweging met snelheid U door een incompressibele vloeistof (dichtheid ρ, viscositeit η), beschreven door de Oseenvergelijking, kan voor kleine waarden van het Reynoldsgetal $R = \rho Ua/\eta$ gegeven worden in de vorm van een machtreeks in R. Deze machtreeks heeft een nulpunt bij $R = -1,466..$.

(2) De benadering voor de 'Oseen' drag coefficient van een bol (zie stelling 1), gebaseerd op de eerste drie krachtmultipolen, levert voor $R = \infty$ een waarde op die ongeveer 10% afwijkt van de door Stewartson berekende asymptotische waarde 3.33 .

K. Stewartson, Phil. Mag. 1 (1956) 345 .
Dit proefschrift, hoofdstuk 2 .

(3) De door Stewartson berekende waarde voor de weerstand die een impulsief gestarte bol ondervindt bij beweging langs de as van een roterende ideale vloeistof als alle aanloopverschijnselen zijn uitgedempt, kan ook op eenvoudige, exacte wijze worden afgeleid met de methode van geïnduceerde krachten.

K. Stewartson, Proc. Camb. Phil. Soc. 48 (1952) 168 .
Dit proefschrift, hoofdstuk 4 .

(4) De geïnduceerde ladingsdichtheden op twee geleidende ongeladen bollen die zich in een homogeen electrische veld bevinden kunnen met behulp van een multipoolontwikkeling worden berekend. Voor het geval dat het veld evenwijdig is aan het lijnstuk dat de centra der bollen verbindt, en dat de bollen elkaar raken, schijnt de numeriek berekende waarde van het dipoolmoment als functie van het aantal in rekening gebrachte multipolen logarithmisch te divergeren. Het is evenwel mogelijk om door middel van een afschatting aan te tonen dat het dipoolmoment begrensd is.

(5) De kracht, die wordt uitgeoefend door een geluidsgolf met frequentie ω op een zich in een vloeistof bevindende cilinder bij inval loodrecht op de as van de cilinder, is voor voldoend hoge, 'subsonische' frequenties evenredig met $\omega^{-\frac{1}{2}}$. Onder subsonische frequenties worden frequenties verstaan waarbij de waarde van de geluidsnelheid in de vloeistof - die frequentie-afhankelijk is - in goede benadering gelijk is aan de waarde bij $\omega = 0$.

(6) Het verdient aanbeveling om voor soortelijke warmte metingen bij lage temperaturen als standaardpreparaat zilver te gebruiken in plaats van koper.

(7) Voor de relatieve verschuiving G van de waarde van B/p (B de magnetische veldsterkte en p de druk) als functie van de dichtheid waarbij het transversale viscomagnetische effect maximaal is, kan de volgende uitdrukking worden afgeleid:

$$G = 1 + 3\sqrt{2}(4nb\,\mathfrak{S}(02\pi)\,)^{-1}$$

Hier is n de dichtheid, b de breedte van het kanaal en $\mathfrak{S}(02\pi)$ de effectieve vervalsdoorsnede voor $\underline{\underline{\phi}}^{02\pi}$ - polarisatie.

(8) Gegeven drie, naar oppervlaktemaat uniform verdeelde, willekeurig gekozen punten op een bol. De waarschijnlijkheid dat de vlakke driehoek, die gevormd wordt door de lijnstukken die deze drie punten verbinden, een stompe hoek bevat is gelijk aan $\frac{1}{2}$.

Probleem ontleend aan Mathematical Intelligencer, vol. 7, nr. 2, 1985 .

(9) Het voorkomen van grote aantallen restanten van vaak eeuwenoude kleien pijpen, verspreid over gebieden met oppervlakten van zeer vele vierkante kilometers, kan op eenvoudige wijze worden verklaard.

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