

## A RIGOROUS NONPERTURBATIVE RESULT FOR THE GLUEBALL MASS AND ELECTRIC FLUX ENERGY IN A FINITE VOLUME

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We present the final result for the short-distance expansion of the energy of 't Hooft-type electric flux, in units of the glueball mass. As a spin-off, we also give the nonperturbative correction for the short-distance expansion of the glueball mass and mass ratios, which turn out to be already significant at  $z = M(0^+)L = 1$ . We propose a way to bridge the gap between the short-distance expansion ( $z < 1$ ) and the Monte Carlo data ( $z > 1.5$ ). We provide evidence by reinterpreting existing Monte Carlo data, and discuss the nature of the  $Z_2$  symmetry-breaking transition.

### 1. Introduction

The main result of this paper is the final numerical evaluation of the constants in the short-distance expansion for the energy of electric flux in a pure  $SU(2)$  gauge theory on the torus  $T^3$ . (The fields are periodic with period  $L$  in the 3 spatial directions.) In his pioneering work 't Hooft [1] showed how to give a gauge-invariant meaning to nonabelian electric (and magnetic) flux, using twisted boundary conditions. It is for this electric flux, and for a nonperturbative glueball mass correction that we derive a rigorous short-distance result. A very important and perhaps crucial advantage of this approach is the absence of infrared divergences.

We will outline a few steps in the derivation of the result, but for details see ref. [2] and a forthcoming paper [20]. The method is based on deriving an effective hamiltonian or lagrangian for the zero-momentum modes. We are ultimately interested in the long-distance behaviour, so these zero-momentum or long-wavelength modes are expected to play an important role. This effective theory is well defined for small  $L$ , since the momenta behave as  $2\pi n/L$ , with  $n_i$  integer, and the zero-momentum sector is well separated in energy from the nonzero-momentum sectors.

In the effective hamiltonian approach special attention is paid to a quantum-induced potential barrier. Tunneling through this barrier removes an 8-fold degener-

acy labeled by electric flux. For the ground state the energy split gives a nonzero value to the energy of electric flux. It will become clear that tunneling for the excited levels arises through the same “channel” as for the ground state, and in weak coupling only deviates from the result for the ground state through its dependence on the energy and its coupling to this dominant tunneling mode. It implies that both the glueball mass and the mass ratios are affected by tunneling which sets in for  $z \equiv M_L(0^+)L < 1$ .

Beyond  $z = 1$  perturbative and semiclassical approximations for the energy levels are inapplicable, but the effective hamiltonian itself might very well be trusted up to  $z = 2$ , or at least describe the main features of the dynamics. Arguing from this we suggest a way to close the gap between the short-distance expansion and the Monte Carlo results. Indeed, Monte Carlo results for the same quantities we consider have been obtained recently by Berg and Billoire [10]. Reinterpreting their results and combining them with our short-distance expansion almost closes this gap.

This paper is organized as follows. In the next section we describe the effective hamiltonian and the quantum-induced potential barrier. Sect. 3 describes the tunneling ingredients and sect. 4 fills in the numerical values. Sect. 5 describes how to bridge the gap and provides circumstantial evidence by comparing our result and existing Monte Carlo data. In this section we also show how to calculate, in principle, the value of the lattice coupling at which a crossover in the Monte Carlo evaluation of the expectation value for a spatial Polyakov loop occurs. In sect. 6 we ask ourselves whether magnetic flux can be of any help, and we end with conclusions and a plea to the Monte Carlo community.

A reader mainly interested in the results is advised to start at sect. 4, which is more or less self-contained.

## 2. Effective hamiltonians and all that

Let us first briefly review the shape of the classical potential:  $V = \frac{1}{4} \int \text{Tr}(F_{ij})^2$ . The potential is identically zero for vector potentials which, up to a periodic gauge transformation, are spatially constant and abelian (i.e., diagonal). We label these by  $C$  through  $A = (C\sigma_3/2L)$ .  $C$  parameterizes what we call a vacuum valley [2], but the shape of the potential in other directions depends on  $C$ . This dependence has a periodicity of  $2\pi$ , because the translation of  $C$  over  $2\pi e_i$  is generated by an allowed gauge transformation, satisfying  $\Omega(x + Le_i) = (-1)^{k_i} \Omega(x)$ . Hence we only need to know the effective hamiltonian in one unit cell of  $C$ , ( $|C_i| \leq \pi$ ). A wave functional is invariant under the above gauge transformations, only up to a phase factor  $\exp(\pi e_i k_i)$ , where  $e \in (\mathbb{Z}_2)^3$  is the nonabelian electric flux ( $k \in (\mathbb{Z}_2)^3$  is the hamiltonian equivalent of a twist in the time direction [1]).

The wave functional will be concentrated around points where  $V$  is widest, which is at  $C/2\pi \in \mathbb{Z}^3$ . Here something dramatic occurs, first discussed in detail for the lattice in connection with the so-called groundstate metamorphosis [3]. Generically,

$V$  is quadratic for variations perpendicular to the vacuum valley, but at  $C = \mathbf{0}$  the potential is quartic in *all* constant vector potentials:  $V = (-L^3/4g^2)\text{Tr}([A_i, A_j]^2)$ . These quartic modes reoccur for all  $C/2\pi \in \mathbb{Z}^3 \setminus \{0\}$ , where they are related to constant modes by the same gauge transformations which map  $C = \mathbf{0}$  to  $C/2\pi \in \mathbb{Z}^3 \setminus \{0\}$ . This quartic mode problem is a topological feature which cannot be removed by choosing a different metric on the torus, and occurs in one form or another in all approaches to this subject. It can be regarded as a manifestation of the infrared problem, and prevents the naïve application of perturbation theory. (One could however get rid of these quartic modes by introducing magnetic flux, but at the cost of other complications. We will comment on this in sect. 6.)

Intuitively one would like to derive an effective hamiltonian for the three parameters  $C_i$  of the vacuum valley, which we will denote by  $H_{\text{valley}}$ . This would be obtained by integrating out all other degrees of freedom. However, because of the quartic-mode problem, a simple loop expansion for  $H_{\text{valley}}$  breaks down near the points  $C/2\pi \in \mathbb{Z}^3$ . Lüscher [4] resolved this quartic-mode problem by deriving an effective hamiltonian for all zero-momentum modes, i.e. those vector potentials  $A_i$  which satisfy  $\partial_i A_j + i[C_i \sigma_3/2L, A_j] = 0$ . For  $C = \mathbf{0}$  this is equivalent to deriving a nine-dimensional effective hamiltonian in the spatially-constant vector potentials, labeled by  $c/L$ , which we will call  $H_{\text{pert}}$ .

Thus, provided we stay far enough away from  $C/2\pi \in \mathbb{Z}^3$ , an effective hamiltonian in terms of  $C$  is easily derived to be:

$$H_{\text{valley}} = -\frac{g^2}{2L} \frac{\partial^2}{\partial C^2} + V_1(C), \quad (1)$$

where  $g$  is the *renormalised* coupling constant and  $V_1$  the one-loop effective potential [2, 4], periodic in  $C$  with periods  $2\pi$  in all 3 directions (see eq. (A.4)).  $V_1$  has the property that it is minimal along the coordinate axes and behaves as

$$V_1(C) = \frac{2|C|}{L} - \frac{(0.30104661\dots)}{L} C^2 + O(C^4). \quad (2)$$

It is only in fourth order in  $C_i$  that the spherical symmetry in  $V_1$  is broken, where it gets replaced by the cubic symmetry  $O(3, \mathbb{Z})$ .

On the other hand, for  $C/2\pi$  close to integer values one can use Lüscher's effective hamiltonian [4], which in lowest order is ( $A = c/L$ ):

$$H_{\text{pert}} = -\frac{g^2}{2L} \frac{\partial^2}{\partial c_i^2} - \frac{1}{2g^2 L} \text{Tr}[c_i, c_j]^2, \quad (3)$$

i.e., just the Yang-Mills hamiltonian for a constant vector potential, but with a renormalised coupling constant. This hamiltonian exhibits intricate dynamics, and is

interesting in its own right. It was recently extensively studied by Savvidy [5] who showed, amongst other things, that this hamiltonian is nonintegrable. This fact was conjectured independently by Lüscher and Münster [6], and means one must resort to numerical techniques to solve for the energy eigenvalues, for which they used Rayleigh-Ritz perturbation theory.

Wave functionals are only strictly periodic for translations of  $C_i$  over  $4\pi$ , hence there exist  $2^3$  inequivalent vacua:  $C/2\pi \in \{0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\}$ . Since in perturbation theory one can expand around any one of the  $2^3$  inequivalent vacua [4], there is an 8-fold degeneracy, which gets lifted by tunneling through the quantum-induced potential barrier described by  $V_1(C)$  [2]. It is interesting to note that while in perturbation theory the  $Z_2$  symmetry is broken, it gets restored by including the tunneling contributions. The tunneling calculation is complicated both by the quartic-mode problem and the fact that the barrier is quantum induced. However, an extensive analysis of toy-models [7] showed how, with the help of the path decomposition expansion [8], one can separately use perturbative quantum mechanics in the well regions and a semi-classical approximation in the barrier region, and then “glue” these contributions to the Green function together to find an asymptotic expansion for the energy splitting, correct up to a relative error growing as a positive power of the coupling constant [7].

### 3. All one need know for tunneling

To understand how one obtains the tunneling result from eqs. (1) and (3) we will rewrite the hamiltonian  $H_{\text{pert}}$  in three steps in such a way that it manifestly reproduces  $H_{\text{valley}}$  for small  $C$ .  $H_{\text{valley}}$  as given in eq. (1) is obtained by an adiabatic approximation for the fluctuations transverse to the vacuum valley. This approximation breaks down close to  $C/2\pi \in \mathbb{Z}^3$ , due to the quartic nature of the potential near these points. Rewriting  $H_{\text{pert}}$  makes clear how one goes beyond the adiabatic approximation and at the same time shows the validity of this approximation for points along the vacuum valley far enough from  $C/2\pi \in \mathbb{Z}^3$ .

The analysis is greatly simplified if we use the spherical symmetry. In that case we can use the polar decomposition [5] for  $SU(2)$  gauge fields (here and henceforth  $x_i$  should not be confused with the space coordinates!)

$$c_i^a = g^{2/3} \xi^{ab} X_{bj} \eta_{ij},$$

$$X = \text{diag}(x_1, x_2, x_3),$$

$$\xi, \eta \in \text{SO}(3). \quad (4)$$

$\eta$  gives the rotational and  $\xi$  the gauge degrees of freedom, and in the gauge-

invariant, spherically-symmetric sector an easy computation shows [5]:

$$H_{\text{pert}} = \frac{g^{2/3}}{2L} \left[ -\frac{1}{J} \frac{\partial}{\partial x_i} J \frac{\partial}{\partial x_i} + (x_1^2 x_2^2 + x_2^2 x_3^2 + x_1^2 x_3^2) \right]. \quad (5)$$

The factor  $g^{2/3}$  in eq. (4) was chosen to make  $H_{\text{pert}}$  proportional to a power of  $g$ . For gauge- and rotation-invariant wave functions,

$$\int d^9 c \Psi(c) \Phi^*(c) = \frac{1}{3} g^6 2\pi^4 \int d^3 x |J| \Psi(x) \Phi^*(x), \quad (6)$$

where  $J$  is the jacobian:

$$J = \prod_{i>j} (x_i^2 - x_j^2). \quad (7)$$

The  $x_i$  are not all gauge and rotationally inequivalent. A permutation and two simultaneous sign changes describe the remaining gauge/rotational freedom. We can hence choose  $x_i$  to uniquely label a rotation- and gauge-invariant state if we restrict it to

$$x_3 \geq x_2 \geq 0, \quad x_2^2 \geq x_1^2. \quad (8)$$

The vacuum valley is now easily identified as the  $x_3$  axis, and we have  $x_3 = g^{-2/3} |C|$ .

As an intermediate step we transform the wave function according to:

$$\tilde{\Psi}(x) = |(x_3^2 - x_1^2)(x_3^2 - x_2^2)|^{1/2} \Psi(x), \quad (9)$$

and for convenience put  $g = L = 1$ . The hamiltonian and inner product now become:

$$\tilde{H} = -\frac{1}{2} \frac{\partial^2}{\partial x_3^2} + \tilde{H}_{[x_3]}, \quad (10)$$

$$\begin{aligned} \tilde{H}_{[x_3]} = & -\frac{1}{2(x_1^2 - x_2^2)} \frac{\partial}{\partial x_i} (x_1^2 - x_2^2) \frac{\partial}{\partial x_i} + \frac{1}{2} x_3^2 (x_1^2 + x_2^2) \\ & + \frac{1}{2} x_1^2 x_2^2 - \frac{1}{2} \frac{x_3^2 + x_1^2}{(x_3^2 - x_1^2)^2} - \frac{1}{2} \frac{x_3^2 + x_2^2}{(x_3^2 - x_2^2)^2}, \end{aligned} \quad (11)$$

$$\int d^3 x |J| \Psi(x) \Phi^*(x) = \int d^3 x |x_1^2 - x_2^2| \tilde{\Psi}(x) \tilde{\Phi}^*(x). \quad (12)$$

From (9)  $\tilde{\Psi}$  satisfies vanishing boundary conditions at  $x_1^2 = x_3^2$  and  $x_2^2 = x_3^2$ . Finally, we solve for the  $(x_1, x_2)$ -dependent eigenfunctions  $\mathcal{X}_{[x_3]}^{(n)}$  of  $\tilde{H}_{[x_3]}$  with energies  $V_n(x_3)$ , where  $n = 1$  labels the ground state. The eigenfunctions are normalised according to:

$$\int_{-x_3}^{x_3} dx_1 \int_{-x_3}^{x_3} dx_2 |x_1^2 - x_2^2| \mathcal{X}_{[x_3]}^{(n)}(x_1, x_2) \mathcal{X}_{[x_3]}^{(m)*}(x_1, x_2) = \delta^{nm}, \quad (13)$$

and are invariant under interchange and reversal of  $x_1$  and  $x_2$  (we can restrict ourselves to positive parity wave functions). This problem can be easily solved in perturbation theory with  $x_3^{-3}$  as expansion parameter. The vanishing boundary conditions for  $\mathcal{X}_{[x_3]}^{(n)}$  at  $x_1^2 = x_3^2$  and  $x_2^2 = x_3^2$  only lead to exponentially small corrections, and can be ignored for  $x_3 \geq 1.5$ . One finds up to first order in  $x_3^{-3}$ :

$$\mathcal{X}_{[x_3]}^{(1)} = \sqrt{\frac{1}{2}} x_3 \left( 1 - \frac{[2x_1^2 x_2^2 x_3^2 + x_3(x_1^2 + x_2^2) - 3]}{16x_3^3} \right) e^{-x_3(x_1^2 + x_2^2)/2},$$

$$V_1(x_3) = 2x_3 - \frac{3}{4x_3^2}. \quad (14)$$

$\tilde{H}$  can now be expressed w.r.t. the following expansion of the wave function [7]:

$$\tilde{\Psi}(x_1, x_2, x_3) = \sum_{n=1}^{\infty} \Phi^{(n)}(x_3) \mathcal{X}_{[x_3]}^{(n)}(x_1, x_2), \quad (15)$$

as

$$H_{nm} \Phi^{(m)} = -\frac{1}{2} \left( \delta_{nm} \frac{\partial}{\partial x_3} - A_{nm}(x_3) \right)^2 \Phi^{(m)}(x_3) + V_n(x_3) \Phi^{(n)}(x_3), \quad (16)$$

where the “gauge field”  $A$  is given by

$$A_{nm}(x_3) = \int_{-x_3}^{x_3} dx_1 \int_{-x_3}^{x_3} dx_2 |x_1^2 - x_2^2| \mathcal{X}_{[x_3]}^{(m)}(x_1, x_2) \frac{\partial}{\partial x_3} \mathcal{X}_{[x_3]}^{(n)*}(x_1, x_2). \quad (17)$$

It is also straightforward to repeat the careful analysis of ref. [7]. One finds that  $\Phi^{(n)}(x_3)/\Phi^{(1)}(x_3)$  decreases at least as  $x_3^{-3/2}$ , but often exponentially with  $x_3$ . Hence for  $x_3^2 \gg (x_1^2, x_2^2)$ ,  $\tilde{\Psi}(x_1, x_2, x_3) = \Phi^{(1)}(x_3) \mathcal{X}_{[x_3]}^{(1)}(x_1, x_2)$ . In this sense,  $\Phi^{(1)}(x_3)$  is the dominant “channel”, i.e. tunneling mode. Indeed, in the barrier region, this component gives the dominant contribution to the Green function (c.f.

[7]). For  $x_3$  sufficiently large,  $\Phi^{(1)}(x_3)$  satisfies the Schrödinger equation:

$$\left( -\frac{1}{2} \frac{\partial^2}{\partial x_3^2} + 2x_3 - \frac{1}{2x_3^2} + O(x_3^{-5}) \right) \Phi^{(1)}(x_3) = \varepsilon \Phi^{(1)}(x_3), \quad (18)$$

where  $E = g^{2/3}\varepsilon/L$  is the lowest order (in  $g^{2/3}$ ) energy of the ground state for  $H_{\text{pert}}$ . The potential of eq. (18) is the sum of  $V_1(x_3)$  and  $A_{0m}(x_3)A_{m0}(x_3)$ . Reinstating the proper  $g$  and  $L$  dependence, (i.e. multiplying eq. (18) by  $g^{2/3}/L$ ) the potential of eq. (18) becomes

$$\frac{g^{2/3}}{L} \left( 2x_3 - \frac{1}{2x_3^2} + O(x_3^{-5}) \right). \quad (19a)$$

Putting  $|C| = g^{2/3}x_3$  in eq. (2) gives

$$\frac{g^{2/3}}{L} \left( 2x_3 - (0.30104661 \dots) g^{2/3}x_3^2 + O(x_3^4) \right). \quad (19b)$$

Thus both potentials agree for  $g^{-2/3} \gg x_3 \gg 1$ , which is essential for the consistency of the approach. This consistency requirement is discussed further in appendix A.

Looking at the toy-models of ref. [7],  $g^{-2/3} \gg x_3 \gg 1$  is exactly the region where we “glue” the perturbative wave function to the tunneling contribution (the transition Green function).

It turns out that all one needs from the perturbative region is the energy eigenvalue and the asymptotic behaviour of the wave function for  $|C| \gg g^{2/3}$ . Were we to take  $H_{\text{valley}}$  as given in eq. (1) to calculate the energy splitting, the asymptotic behaviour of its perturbative wave function  $\tilde{\Phi}(C)$  would be wrong by a constant factor, and its energy eigenvalue  $\tilde{\varepsilon}$  would not equal  $\varepsilon$ . This is because  $H_{\text{valley}}$  is an inadequate approximation in the perturbative region. However, one can calculate the energy of electric flux by using  $\varepsilon$  instead of  $\tilde{\varepsilon}$  and  $\Phi^{(1)}$  instead of  $\tilde{\Phi}$  in the expression for the energy splitting as derived from  $H_{\text{valley}}$ .

Equivalently, one could change  $V_1(C)$  in the perturbative region ( $|C| < g^{2/3}$ ) such that it does give the correct energy eigenvalue and asymptotic behaviour of the wave function. In that case, the modified version of  $H_{\text{valley}}$  gives an expression for the energy splitting identical to the energy of electric flux, up to relative errors which grow as positive powers of the coupling constant.  $H_{\text{valley}}$  is very similar to the double cone problem studied in ref. [7], and from this we easily read off the expression for  $\Delta E$  ( $C^*$  is the classical turning point, accidentally omitted in ref. [2b], eq. (14)):

$$\Delta E = 2\lambda |B|^2 g^{5/3} L^{-1} \exp \left( -\frac{1}{g} \int_{C^*}^{2\pi - C^*} \sqrt{2(LV_1(Ce_1) - g^{2/3}\varepsilon)} dC \right), \quad (20)$$

where  $\lambda$  is a constant determined by the transverse fluctuations [7,20] and  $B$  is given by:

$$B = \lim_{r \rightarrow \infty} (4r - 2\varepsilon)^{1/4} \exp\left(\frac{1}{6}(4r - 2\varepsilon)^{3/2}\right) \Phi^{(1)}(r). \quad (21)$$

In this equation  $B$  is  $g$ -independent, therefore one has to normalise the wave-function  $\tilde{\Phi}(C)$  for the purely three-dimensional problem according to  $g^{-2} \int d^3C |\tilde{\Phi}(g^{-2/3}C)|^2 = 1$ . If we express  $\Phi^{(1)}$  in terms of the ground state wave function  $\Psi(x_1, x_2, x_3)$ , we easily deduce from this that  $\Psi$  has to be normalised as:

$$1 = 4\pi \int_0^\infty dx_3 \int_{-x_3}^{x_3} dx_1 \int_{-x_3}^{x_3} dx_2 |J| |\Psi(x)|^2 = g^{-6} \pi^{-3} \int d^9c |\Psi(c)|^2. \quad (22)$$

We have used eq. (6) and the invariance of  $\Psi$  under permutations and reflections of the coordinates for the last part of this identity. Hence we derive the following two equivalent expressions for  $\Phi^{(1)}$  (see eq. (15)):

$$\begin{aligned} \Phi_I^{(1)}(x_3) &= \pi^{3/2} N \int_{-x_3}^{x_3} dx_1 \int_{-x_3}^{x_3} dx_2 |x_1^2 - x_2^2| \\ &\quad \times \left[ (x_3^2 - x_1^2)(x_3^2 - x_2^2) \right]^{1/2} \Psi(x) \mathcal{X}_{[x_3]}^{(1)}(x_1, x_2), \\ \Phi_{II}^{(1)}(x_3) &= \pi^{3/2} N \Psi(x_1 = x_2 = 0, x_3) x_3^2 / \mathcal{X}_{[x_3]}^{(1)}(0, 0) \quad (\text{for } x_3 \gg 1), \end{aligned} \quad (23)$$

where

$$N^{-2} = g^{-6} \int d^9c |\Psi(c)|^2. \quad (24)$$

For  $x_3$  sufficiently large, where  $\Phi^{(n)}$  ( $n > 1$ ) can be neglected w.r.t.  $\Phi^{(1)}$ ,  $\Phi_I$  and  $\Phi_{II}$  will coincide.

Eq. (21) tells us that  $\Phi^{(1)}(r)$  will decay rapidly. We calculated the wave function numerically [20], using Rayleigh-Ritz perturbation theory for  $H_{\text{pert}}$  to a higher accuracy than in ref. [6]. Since convergence of the Rayleigh-Ritz procedure is in the  $L^2$  sense, the *relative* error in  $\Phi^{(1)}(r)$  will rapidly increase with  $r$ . Eq. (21) is therefore not useful for practical purposes. Instead, let us introduce  $\Phi_0(r)$ , which satisfies the same asymptotic equation as  $\Phi^{(1)}$  (eq. (18)), but normalised so as to give  $B = 1$  in eq. (21), i.e.

$$\begin{aligned} \left( -\frac{1}{2} \frac{\partial^2}{\partial r^2} + 2r - \frac{1}{2r^2} \right) \Phi_0(r) &= \varepsilon \Phi_0(r), \\ \lim_{r \rightarrow \infty} (4r - 2\varepsilon)^{1/4} \exp\left(\frac{1}{6}(4r - 2\varepsilon)^{3/2}\right) \Phi_0(r) &= 1. \end{aligned} \quad (25)$$



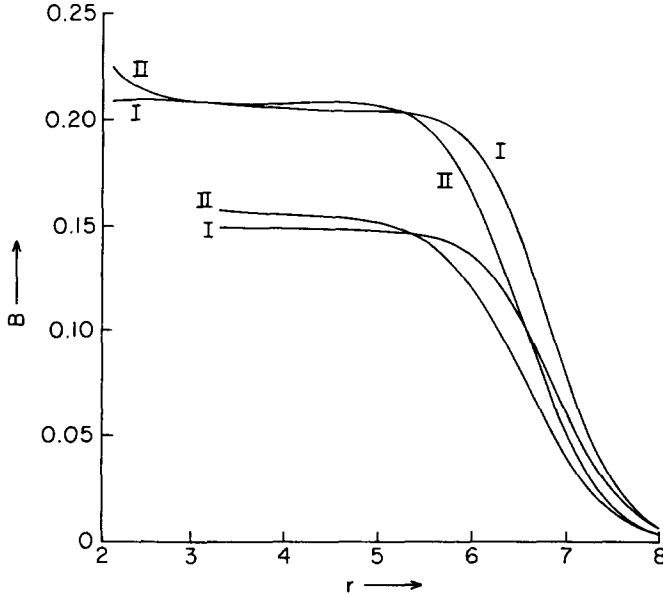


Fig. 1. The results for  $B_I(r)$  and  $B_{II}(r)$ . The upper two curves are for the ground state and the lower two curves for the first excited state. These results were used to extract  $B$  (eq. (27)) and  $B'$  (eq. (32)).

In fig. 1 we plot  $B_{I(II)}(r) = \Phi_{I(II)}^{(1)}(r)/\Phi_0(r)$ . We determined  $\Phi_0(r)$  by numerical integration and verified that including an  $O(r^{-5})$  term in the potential of the Schrödinger equation for  $\Phi_0(r)$  yields a relative change in  $\Phi_0(r)$  smaller than 0.5% for  $r > 2.5$ . For the ground state we see from fig. 1 that  $B_I$  and  $B_{II}$  coincide for  $r > 2.9$  within 0.2% accuracy. That they deviate beyond  $r = 3.6$  is due to numerical errors in the wave function bigger than 0.2%. We can therefore reliably extract  $B$  from  $B_{I(II)}$  around  $r = 3.4$ . The introduction of  $\Phi_0(r)$  is indeed necessary, since  $(4r - 2\epsilon)^{1/4} \exp(\frac{1}{6}(4r - 2\epsilon)^{3/2}) \Phi_0(r) \sim 0.95$ , for  $r \sim 3.4$ , which differs significantly from 1.

#### 4. The numerical results for the semiclassical parameters

The energy of electric flux  $\Delta E(L)$ , eq. (20), was previously calculated up to a constant prefactor [2]:

$$\Delta E = A g^{5/3} L^{-1} \exp(-Sg^{-1} + T\epsilon g^{-1/3})(1 + O(g^\gamma)),$$

$$S = 12.4637\dots, \quad T = 3.9186\dots, \quad \frac{1}{6} < \gamma \leq \frac{1}{3}. \quad (26)$$

The ground state energy eigenvalue  $\epsilon = 4.11672\dots$  was numerically evaluated by

Lüscher and Münster [6]. As discussed in sect. 3, we repeated their Rayleigh-Ritz procedure to a higher accuracy in order to evaluate the wave function and thus fix the prefactor  $A = 2\lambda|B|^2$ . Also,  $\lambda$  had to be numerically evaluated. In a separate publication we will describe the calculations in more detail [20], here we only give the results:

$$\begin{aligned}\lambda &= 0.69970 \pm 0.00001, \\ B &= 0.2063 \pm 0.0008.\end{aligned}\tag{27}$$

This value of  $B$  was extracted from the plateau in fig. 1.

To get a renormalization group invariant expression we express

$$\mathcal{E}(L) = \Delta E(L)/M_L(0^+)\tag{28}$$

in terms of the universal expansion parameter  $z$  [9]:

$$z = M_L(0^+)L = 2.2696390g^{2/3} - 0.7975278g^{4/3} + O(g^2),\tag{29}$$

and substitute for  $\varepsilon$  the ground state energy (more precisely  $E = g^{2/3}\varepsilon/L$ ):

$$\varepsilon = 4.116719735 - 1.174516027g^{2/3} + O(g^{4/3}).\tag{30}$$

(The coefficients in eq. (29) and (30) were first evaluated in ref. [6], but can easily be obtained to a higher accuracy from our own numerical analysis.) One finds:

$$\mathcal{E}(z) = 0.00767z^{3/2}\exp(-42.6169z^{-3/2} + 34.2001z^{-1/2}).\tag{31}$$

The possible error in the prefactor is 0.8%, and the relative error due to the semiclassical approximation is at most  $O(z^{1/4})$  (and most likely  $O(z^{1/2})$ , obtained from evaluating the integral in the exponent of eq. (20) to a higher order, leading to a multiplicative factor of approximately  $\exp(4z^{1/2})$ ).

We can strictly trust eq. (31) only for  $z \ll 1$ . In a previous paper [2b], one of us expressed the hope that one can trust eq. (31) all the way up to  $z \sim 3$ , where it has a remarkably linear behaviour, which would fit to an expected linear long-distance behaviour of  $\mathcal{E}(z)$ . With the actual value of the prefactor we have now computed, this turns out to be unrealistic. Also the tunneling sets in between  $z = 1.1$  and  $z = 1.3$  (instead of  $z \sim 2$ ), where  $\mathcal{E}(z)$  grows from  $< 0.0001$  to  $> 0.05$ . It furthermore implies that the perturbative result obtained by Lüscher and Münster for  $M(0^+)/\Lambda_{\overline{\text{MS}}}$  as a function of  $z$  will be drastically modified for  $z > 1$ .

The weak coupling expansion for the shift in the glueball mass due to the tunneling can be easily evaluated by calculating the energy split  $\Delta E'(L)$  for the first excited level, which is obtained by simply repeating the analysis for the ground state but replacing  $\Psi$  and  $\varepsilon$  by  $\Psi'$  and  $\varepsilon'$ , the numerically evaluated first excited wave

function and energy. We find:

$$\begin{aligned}\epsilon' &= 6.3863588 - 1.9720438g^{2/3} + O(g^{4/3}), \\ B' &= 0.1480 \pm 0.0008,\end{aligned}\tag{32}$$

whereas  $\lambda$ ,  $S$  and  $T$  are not affected\*. Hence we can calculate  $\mathcal{E}'(L) = \Delta E'(L)/M_L(0^+)$  and express it in terms of  $z$ :

$$\mathcal{E}'(z) = 0.00395z^{3/2}\exp(-42.6169z^{-3/2} + 47.5989z^{-1/2}).\tag{33}$$

In weak coupling the energy of an  $e=0$  level will shift downwards by half the energy difference with the  $|e|=1$  state. Hence the relative mass shift is:

$$\frac{\Delta M_L(0^+)}{M_L(0^+)} \cong (\mathcal{E}(z) - \mathcal{E}'(z))/2 \cong -\frac{1}{2}\mathcal{E}'(z).\tag{34}$$

We need not stop here, and can evaluate similarly  $\mathcal{E}^{(\nu)}(L)$  for the  $\nu$ th excited level, which gives the following result for the shift in mass ratios:

$$\begin{aligned}\Delta\left(\frac{M_L^{(\nu)}(0^+)}{M_L(0^+)}\right) &\cong -\frac{1}{2}\mathcal{E}^{(\nu)}(z), \\ \mathcal{E}^{(\nu)}(z) &= 2\lambda|B^{(\nu)}|^2z^{3/2}\exp(-42.6169z^{-3/2} + (9.897 + 5.903\epsilon^{(\nu)})z^{-1/2}).\end{aligned}\tag{35}$$

Thus all mass ratios will be affected significantly by the tunneling. Furthermore, the larger the energy  $\epsilon^{(\nu)}$  the earlier the tunneling sets in (for obvious reasons); for the energy of electric flux it sets in around  $z=1.2$ , for the glueball mass around  $z=1$  and for the mass ratios at even smaller values of  $z$ .

In ref. [6] three “obvious questions” were posed:

- (a) At what  $z$  does tunneling effectively set in?
- (b) Is there a direct connection with the crossover from small- to large-volume behaviour of the masses?
- (c) Are the mass ratios in the zero “electric” flux sector affected by the tunneling?

Of these questions we have now settled (a) and (c). The fact that tunneling sets in so early fits extremely well with the conclusion of Berg and Billoire [10], using Monte Carlo calculations, that the perturbative result [6] for  $M_L(0^+)$  cannot be used for  $z \sim 1.6-2$ , which unfortunately is about the value of  $z$  that Monte Carlo calculations can reach with some confidence.

\* For the first excited state there is a much larger fraction of the wave function contained in  $\Phi^{(n)}$  ( $n > 1$ ), signalled by a deviation of  $B'_I$  and  $B'_{II}$ , and we have to resort to  $B'_I$  alone to extract  $B'$  (see fig. 1).

We will comment on (b) in the next section. Recently, this point was discussed in detail for the two-dimensional  $O(3)$  model [24], where the Monte Carlo data seem to fit satisfactorily to the short-distance expansion for the mass gap. However, that model is not plagued by the tunneling phenomenon. There the only difficulty is that one has to go beyond the two-loop beta function to establish scaling for the mass gap. We avoid this problem by looking at a dimensionless quantity.

## 5. How to bridge the gap, $z = 1$ to $z = 2$

### 5.1. THEORETICAL DISCUSSION

We want to present what we think is strong circumstantial evidence for the existence of a reliable short- to long-distance connection, provided one is willing to leap the hurdle of the (induced) potential barrier. We first observe that there can be two physically distinct crossovers, which could easily confuse this issue, so let us clearly disentangle them. First, there is possibly (but not necessarily) a crossover from weak to strong coupling, due to the renormalization group, or the flow of the coupling constant. This affects all quantities with the dimensions of a mass equally strongly; Monte Carlo data suggest it occurs around  $z = 3$ . On the other hand, there is the “crossover” due to the onset of tunneling, which is in principle unrelated to the first crossover and occurs for  $z < 1$  depending on what quantity we consider. This “crossover” has now lost its mystery. Whereas the first crossover should not affect dimensionless quantities expressed in terms of the renormalization group invariant parameter  $z$ , the second “crossover” does affect these quantities, as we showed in detail. However, it is reasonable to assume that we can trust the effective hamiltonian (provided we construct it to a high enough order in  $g$ ) in the way described in ref. [2], all the way up to  $z = 2$  or even beyond. There is however a fundamental obstacle to pushing this to high values of  $z$ . The zero momentum states should not lose their identity and hence  $M_L(0^+)$  should stay smaller than the energy ( $\sim 2\pi/L$ ) of the first nonzero momentum state or equivalently  $z < 2\pi$ , although this basically depends on how the latter state (“ $J = 1$ ” [6]) behaves for large  $L$ .

We can therefore reliably “climb” the induced potential barrier by numerically evaluating the energies for this effective hamiltonian. This should be no more difficult than a molecular physics problem (which however can still be hard enough).

One believes the dynamics at long distances to be described by a string picture. Almost in contradiction to this, we still expect the long-distance behaviour to be derivable from some sort of an effective hamiltonian in the zero- (or nearly zero-) momentum modes. To reconcile this one has to realise that even in the presence of electric flux, the lowest energy state has zero momentum, in particular in the direction perpendicular to the electric flux. (Even if the flux is generated by colour

sources, for large separations the roughening [14] will set in, so the localised nature of a flux tube completely disappears.)

Before comparing our expressions with existing Monte Carlo data, let us present a short-distance expansion of the heavy quark potential  $V_L(\mathbf{R})$  and review the existing predictions for the long-distance behaviour.

## 5.2. SHORT-DISTANCE EXPANSION FOR THE HEAVY QUARK POTENTIAL

Lüscher's effective hamiltonian is very well suited to a straightforward derivation of a short-distance expansion of the heavy quark potential, which should be useful for Monte Carlo purposes. In appendix B we will derive the following result, which consistently incorporates finite size effects:

$$-\frac{V_L(\mathbf{R})}{M_L(0^+)} = \alpha_1 z + (\alpha_2 + \alpha_3 U(\mathbf{R}/L)) z^2 + O(z^3), \quad (36)$$

with

$$\alpha_1 = 0.2524585 \dots, \quad \alpha_2 = 0.0569698 \dots, \quad \alpha_3 = 0.08553219 \dots, \quad (37)$$

and where  $U(\mathbf{R}/L)$  is the nonzero momentum contribution, which reduces to the usual perturbative result for  $\mathbf{R}/L \ll 1$ :

$$U(\mathbf{r}) = \frac{3}{16\pi^2} \sum_{\mathbf{p} \neq 0} \frac{\cos(2\pi \mathbf{p} \cdot \mathbf{r})}{p^2} \xrightarrow{\mathbf{r} \rightarrow 0} \frac{3}{16\pi^2 |\mathbf{r}|}. \quad (38)$$

Eq. (B.7) of the appendix gives a rapidly converging expression for the momentum sum in eq. (38). Tunneling contributions will start to play a role for  $z > 1$ .

For the lattice, in the euclidean domain, finite size corrections for the Wilson loop due to the zero momentum modes (not only  $p_i = 0$  but also  $p_0 = 0$ ) were previously considered in ref. [12]. Their result is, however, only valid for large temperature and cannot be used to extract the finite size corrections to the heavy quark potential at zero temperature.

## 5.3. LONG-DISTANCE EXPANSIONS

For the long-distance behaviour the following two predictions are available:

$$M_\infty - M_L \sim - \left( \frac{\lambda}{M_\infty} \right)^2 \frac{3}{16\pi M_L L} \exp\left(-\frac{1}{2}\sqrt{3} M_L L\right), \quad (39)$$

where  $\lambda$  is real and related to a certain forward scattering amplitude in the infinite volume theory [13], and  $M$  stands for the mass gap. The second prediction *assumes* the formation of electric flux tubes; the correction to a linear potential is due to the

roughening of the string [14]:

$$V_L(\mathbf{R}) \approx \sigma |\mathbf{R}| + c_0 - \frac{\pi}{12|\mathbf{R}|}, \quad (40)$$

$$\Delta E(L) \approx \sigma L + c'_0 - \frac{\pi}{3L}. \quad (41)$$

Here  $\sigma$  is the string tension. Eq. (40) is obtained by taking a string with fixed end points (the colour sources) and modelling the transverse fluctuations by a 1 + 1 dimensional nonlinear sigma-model [15] (thereby assuming the most obvious “universality class” for the pure QCD string [27]). The Coulomb term is universal, i.e., does not depend on the interactions of the nonlinear sigma-model. Eq. (41) is obtained by taking a closed string (closed by the periodic boundary conditions) and simply repeating the analysis of ref. [15]. Due to the universality, one need not worry about the periodic boundary conditions in the direction transverse to the string, even for spatially elongated boxes. The value of  $\frac{1}{3}\pi$  in eq. (41) is also known from finite temperature corrections to the Polyakov loop in the confining phase [25]. The constants  $c_0, c'_0$  and their difference can easily be shown to be not universal and are hence not predictable from a string picture. A more speculative prediction will be considered at the end of this section.

Let us now discuss what Monte Carlo data are available and how to represent them. First of all the data of ref. [16] reproduce eq. (40) reasonably well. The authors work on a  $15^4$  lattice and eq. (36) is presumably not applicable. Monte Carlo calculations aiming at producing data also for low  $z$ -values were recently performed by Berg and Billoire [10]. They work on lattices elongated in the time direction with sizes up to  $6^3$  in the space directions ( $N_s = 6$ ) and up to 64 in the time direction ( $N_t = 64$ ). Hence they are guaranteed to be at zero temperature. They calculate the mass gap ( $M_L$ ) from time correlations for Wilson loops in the *adjoint* representation. The Wilson loop, running in the space direction, is closed by the periodic boundary conditions. Hereafter we call this type of Wilson loop a spatial Polyakov loop.

Here we should point out a difficulty in identifying to what state the mass gap belongs. In weak coupling the smallest mass is a  $2^+$  state [6]. Previous Monte Carlo results seemed to indicate a  $0^+$  state for the mass gap, almost degenerate with a  $2^+$  state [28] (at large  $z$ -values). But very recent Monte Carlo calculations prefer a  $2^+$  state for the mass gap, with  $M(2^+)/M(0^+) = 1.25$  around  $z = 2$  to 1.1 around  $z = 5$ , and behaving smoothly as a function of  $z$  [29]. The value of 1.25 for this ratio is to good approximation equal to the perturbative result. Yet one cannot expect a completely smooth short- to long-distance connection for this ratio. The actual calculation for the tunneling contribution to  $M(2^+)$  is more involved, but assuming that the  $2^+$  level couples to the same tunneling channel as the  $0^+$  level, tunneling for

the latter will set in slightly earlier than for the former, causing a small dip in the ratio  $M(0^+)/M(2^+)$  around  $z = 1$  to 1.2. The effect will be small because of the near degeneracy.

The data suggest a dip in the graph of the mass gap versus  $z$  around  $z = 3$ . This dip could be due to two effects; the coupling constant could depend nonmonotonically on  $z$ . Or, if we can trust the effective hamiltonian up to  $z \sim 3$ , it can be due to the tunneling effect. As alluded to above, a dip could appear because tunneling for an excited state sets in earlier than for the ground state. As a very naïve example of such a behaviour, one might consider the double harmonic oscillator and compute the difference between energies of the two lowest energy even parity states, which is the analogue of the mass gap (e.g. see [11], fig. (5.4)). It seems likely therefore, that the asymptotic value of  $M_L$  is indeed approached from below, as predicted by eq. (39).

#### 5.4. KNOWN MONTE CARLO RESULTS

The strongest evidence we will present for the feasibility of our program is to compare our results with  $\mathcal{E}(z)$  as obtained from Monte Carlo data. The most direct way of measuring the energy of electric flux is by using twisted boundary conditions. Some years ago this was done by Mack and Pietarinen [17] for square lattices up to a size of  $5^4$ . In ref. [2] a formula was given for extracting the energy of electric flux in lattice units. But we have to do better in order to be able to use the data of ref. [17]. The result in lattice units, up to exponential corrections in  $N_t$ , is (see appendix C):

$$\Delta E_\ell = -\frac{1}{N_t} \ln \left[ \frac{\mu(\beta, N_t)}{-2N_t \partial \Delta E_\ell / \partial \beta} \right]. \quad (42)$$

$\mu$  is the quantity measured in the Monte Carlo simulations:

$$\mu = -\frac{1}{2} \frac{\partial}{\partial \beta} \ln(Z_t/Z_u), \quad (43)$$

with  $Z_t$  the twisted and  $Z_u$  the untwisted partition function. Also,  $\mu$  is the difference between the twisted and untwisted plaquette expectation values, and one therefore needs high statistics to get an accurate result. For the small values of  $N_t$  considered in ref. [17] it is important to use eq. (42), instead of [2]  $\Delta E_\ell = -(1/N_t) \ln[(1/N_t)\mu]$ . Furthermore, one can only trust those Monte Carlo calculations for  $z \gg 1$ , where one can neglect finite temperature effects, which supposedly set in when the value of  $\mu$  starts to decrease as a function of  $\beta$ . Since we will need an estimate of  $\partial \Delta E / \partial \beta$ , we shall first discuss the result of ref. [10].

By measuring the time correlation for spatial Polyakov loops in the *fundamental* representation, instead of the adjoint representation, one obtains the energy of

electric flux, instead of the mass gap<sup>\*</sup>. The “string tension”  $K$ , for a box of size  $L \times L \times L$ , is defined to be  $\Delta E/L$  and therefore  $\sigma = \lim_{L \rightarrow \infty} K$ . Calculating both  $K_\ell$  and  $M_\ell$  for lattices with spatial extent  $N_s^3$  ( $\ell$  stands for lattice units, as in eq. (42)) we find:

$$\begin{aligned} z &= M_\ell N_s, \\ \mathcal{E}(z) &= K_\ell N_s / M_\ell, \\ \sqrt{\mathcal{E}(z)}/z &= \sqrt{K_\ell}/M_\ell. \end{aligned} \quad (44)$$

Strictly speaking, one has to convert the mass gap to  $M(0^+)$  in the definition of  $z$  and  $\mathcal{E}(z)$ . Because of the near degeneracy of  $M(0^+)$  and  $M(2^+)$  this conversion will not significantly modify our results (certainly not within the errors inherent in the Monte Carlo data). Hence this modification will be ignored for the time being. For large  $z$ -values  $M_\ell$  is not available and instead we use values of  $\sqrt{K}$ , given in units of  $\Lambda_{\text{lattice}}$ , and assume  $M_L$  for these large values of  $z$  to be approximately constant (cf. eq. (39)) and equal to [10]:  $M_\infty = (175 \pm 25)\Lambda_{\text{lattice}}$ . From the data we can also deduce  $L$  in units of  $\Lambda_{\text{lattice}}$  ( $L \equiv z'/(3.5\sqrt{K_\ell})$  [10]) and hence calculate  $z = M_\infty L$  and  $\mathcal{E}(z) = KL/M_\infty$ . Note that these quantities are sensitive to a possible lack of asymptotic scaling, whereas eq. (44) does not depend on this ambiguity. The results are displayed in fig. 2. (We have used the  $t=4$  estimates for  $M_\ell$ , where they deviated considerably from the  $t=5$  estimates [10].) Also note that the results appear to fall on a universal curve, indicating close-to-continuum behaviour.

Since we now have data for  $M_\ell$  and  $\Delta E_\ell$  available for different  $\beta$  and  $N_s$ , we can also use the results of ref. [17]. We extract the following two data points [17]: (i)  $N_s = 4$ ,  $\beta = 2.25$ ,  $\mu = 2.6 \pm 0.6$  and (ii)  $N_s = 5$ ,  $\beta = 2.40$ ,  $\mu = 2.7 \pm 1.1$ . From ref. [10] we obtain for these points: (i)  $M_\ell = 1.30 \pm 0.03$ ,  $\partial \Delta E / \partial \beta = -1.3 \pm 0.1$  and (ii)  $M_\ell = 0.82 \pm 0.06$ ,  $\partial \Delta E / \partial \beta = -1.0 \pm 0.1$ . This gives: (i)  $z = 5.2 \pm 0.12$ ,  $\mathcal{E}(z) = 0.27 \pm 0.05$  and (ii)  $z = 4.1 \pm 0.30$ ,  $\mathcal{E}(z) = 0.32 \pm 0.13$ . The result coincides within the errors with the other data points (see fig. 2). It shows that Monte Carlo calculations with twisted boundary conditions are consistent with the values obtained from loop-loop correlations, and that it should be useful to improve on the accuracy with the increased computational power of today.

## 5.5. COMPARISON AND INTERPRETATION OF RESULTS

Let us now study what one can learn from fig. 2. First of all it is clear that the strong  $z$  dependence around  $z = 3$ , obvious both in the data of the mass gap and the string tension as a function of  $z$  [10], disappears in the graph of  $\mathcal{E}(z)$ . This adds credence to our idea about the crossover due to the flow of the coupling constant. Also, the data fit extremely well with our short-distance expansion. Had tunneling really set in around  $z = 2$ , we would have been inconsistent with the Monte Carlo

<sup>\*</sup> This important observation relates 't Hooft's QCD on a torus with twisted boundary conditions to conventional lattice QCD with periodic boundary conditions.



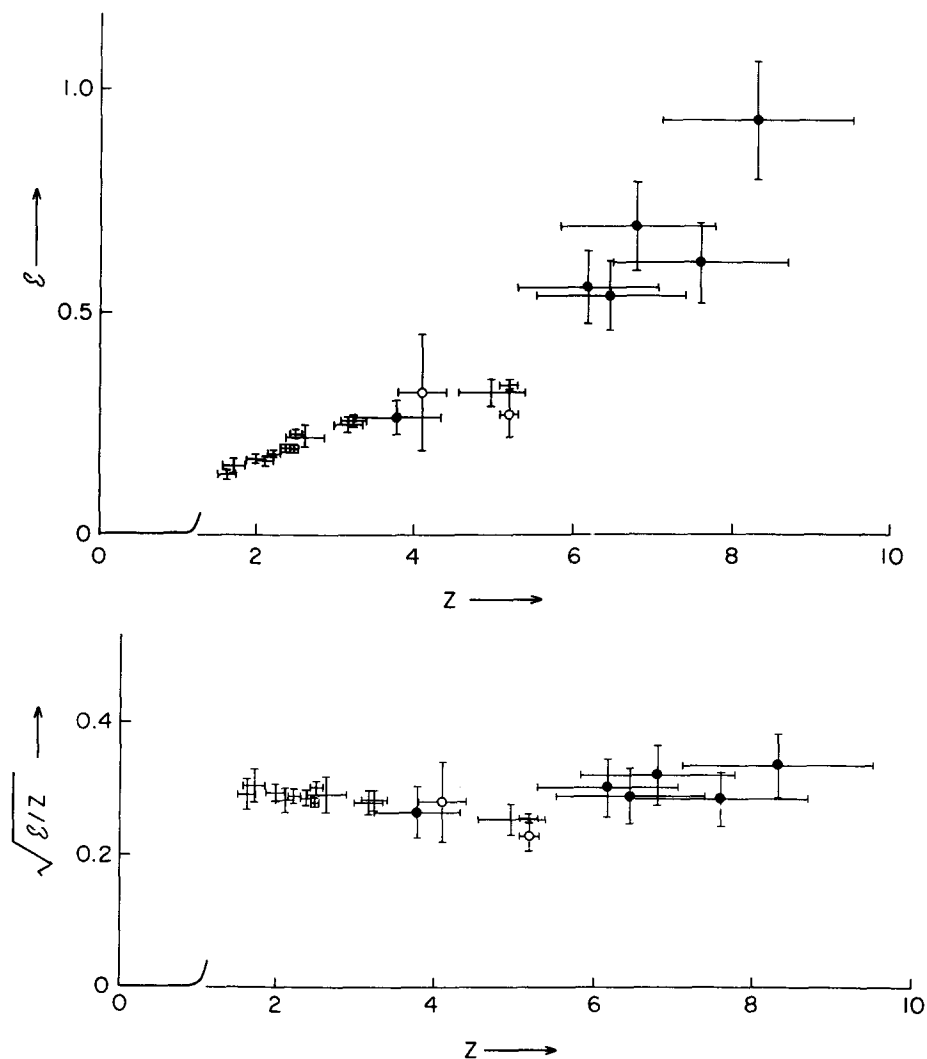


Fig. 2. Results for (a)  $\mathcal{E}(z)$  ( $=\Delta E(L)/M_L$ ), and (b)  $\sqrt{\mathcal{E}(z)/z}$  ( $=\sqrt{K(L)}/M_L$ ). The solid curves represent our short-distance expansion. The Monte Carlo data are specified as follows: + : Independent estimates for  $K_L$  and  $M_L$  from ref. [10], with only statistical errors displayed. (We used the  $\iota=4$  instead of the  $\iota=5$  estimates for  $M_L$  whenever the two disagree significantly.)  $\blacklozenge$  : As above, but using an estimate for  $M_\infty$  of  $(175 \pm 25)\Lambda_{\text{lattice}}$ . Error bars are due to the uncertainty in this quantity.  $\circ$  : Estimates obtained from ref. [17]; error bars are statistical. We ignore the fact that the mass gap could be a  $2^+$  state [29].

data. Indeed, we see no obstacle to taking the effective hamiltonian (including the tunneling domain [2, 20]) and calculating  $\mathcal{E}(z)$  numerically for  $z = 1$  up to at least  $z \sim 2$  and perhaps even beyond. It is important to note that although this is a numerical calculation, it is based on the continuum field theory. This can be an important starting point in reliably working our way out to larger values of  $z$ , and should provide sufficient confidence in Monte Carlo calculations.

The Monte Carlo data can at present reach  $z \sim 1.5$ . For lower  $z$  values it becomes increasingly hard to calculate energies from time correlations. There is also a technical limitation due to the use of a discrete subgroup of  $SU(2)$  in Monte Carlo calculations [29]. We can also give a definite prediction for the behaviour of the expectation value of the spatial Polyakov loop in the fundamental representation ( $\langle P^f \rangle$ ,  $P^f(A) = \frac{1}{2} \text{Tr}(P \exp(i \int_0^L A \cdot d\mathbf{x}))$ ). For  $z$  values where tunneling is suppressed, the Monte Carlo simulations get stuck in one of the perturbative vacua. In other words, the  $Z_2$  symmetry is broken for  $z \rightarrow 0$ , and metastable states occur with  $\langle P^f \rangle = \pm 1$ . The Monte Carlo measurements will therefore exhibit a strong cross-over in  $|\langle P^f \rangle|$  around  $z = 1.2$ , where tunneling sets in, and beyond which  $|\langle P^f \rangle|$  should drop to zero. There is an obvious similarity to the deconfining transition, conjectured to be first order. However, the transition discussed above is smooth. Given  $N_s$ , the spatial size of the lattice, we can in principle predict at what  $\beta$  this transition occurs. If  $\hat{a}(\beta)$  is the two-loop scale factor (which is universal):

$$\hat{a}(\beta) = \left( \frac{6\pi^2\beta}{11} \right)^{51/121} \exp\left( \frac{-3\pi^2\beta}{11} \right), \quad (45)$$

then for the continuum this means we have [6]:

$$L = \Lambda_{\overline{\text{MS}}}^{-1} \hat{a} \left( \frac{4}{g^2(z)} \right) (1 + O(g^2(z))), \quad (46)$$

and for the lattice ( $\beta \equiv 4/g_0^2$ ), using the definition of  $\Lambda_{\text{lattice}}$  [23] and the lattice spacing  $a = L/N_s$ :

$$L = \Lambda_{\text{lattice}}^{-1} N_s \hat{a}(\beta) (1 + O(\beta^{-1})). \quad (47)$$

Were we to have control over the higher order corrections in eqs. (46) and (47), and to  $z(g^2)$  in eq. (29), we could combine eq. (46) and eq. (47), with  $\Lambda_{\overline{\text{MS}}} = 7.462 \Lambda_{\text{lattice}}$ , to calculate  $\beta_c$  corresponding to  $z = 1.2$ . ( $N_s$  should be large enough to suppress lattice artefacts.) However, in the mean time, a Monte Carlo determination of  $\beta_c(N_s)$  will give us some interesting information.

Let us finally take a closer look at the larger  $z$  behaviour in fig. 2. The data suggest, (but not conclusively) that  $\sqrt{\mathcal{E}(z)}/z$  has a dip around  $z = 5$ . Since for the data at  $z > 5$  there was no independent estimate for the mass gap available, this

could be due merely to a systematic error. However if indeed  $\mathcal{E}(z)/z$  is asymptotically constant, the dip would indicate that this asymptotic value is approached from below. The reality of a dip would, however, also imply that a string picture breaks down below  $z = 5$ , which is a bit disappointing. But we'll have to wait and see how far the effective hamiltonian calculation can be pushed to see if a real understanding of confinement can be obtained. Nevertheless fig. 2 does strongly suggest the existence of three distinct domains: (I) The perturbative domain from  $z = 0$  to  $z = 1$ , where the  $Z_2$  symmetry is broken, (II) an intermediate domain where the  $Z_2$  symmetry gets restored by the onset of tunneling, and where the glueball mass is formed, from  $z = 1$  to  $z = 5$ , and (III) a string-like domain, truly nonperturbative, beyond  $z = 5$ . Note that the second domain is probably the most important for the calculation of physical quantities.

Regarding the breakdown of the string picture, it is interesting to note that if we take the Nambu-Goto string at face value, one can calculate  $\mathcal{E}(z)$  exactly [26, 27]:

$$\mathcal{E}(z) = \frac{\sigma z}{M_L^2} \left( 1 - \frac{z_c^2}{z^2} \right)^{1/2}, \quad (48)$$

with

$$z_c^2 = \frac{2\pi}{3} \frac{M_L^2}{\sigma}. \quad (49)$$

The string picture then necessarily breaks down for  $z < z_c$ , and using our value for  $M_\infty^2/\sigma$  gives  $z_c = 4.6$ , which coincides more or less with the dip in fig. 2. It makes it worthwhile to concentrate some Monte Carlo effort on this region, which is easily accessible.

## 6. Can magnetic flux close the gap?

Magnetic flux is introduced by twisted boundary conditions [1]. As is well known from the twisted Eguchi-Kawai model [18], choosing a nonzero twist will make all modes quadratic. Similarly, for the hamiltonian approach we have considered in the continuum, it is easy to show that the presence of magnetic flux removes the vacuum valley, and makes all modes quadratic. Let us take the magnetic flux  $\mathbf{m} = (1, 0, 0)$ . Then [30], both the glueball mass and the energy of electric flux orthogonal to  $\mathbf{m} (\mathbf{e} \cdot \mathbf{m} = 0)$  will to lowest order in  $g$  be proportional to  $1/L$ , whereas the energy of the other electric fluxes will be suppressed by a factor  $\exp(-4\pi^2/g^2)$  [19]. Then, if confinement is realised, 't Hooft's duality arguments [1] tell us to expect that neither the glueball mass nor the string tension will depend on  $\mathbf{m}$ . It would thus be interesting to repeat all the perturbative calculations for this nonzero magnetic flux sector, but although at first sight it seems that this will be much easier due to the absence of quartic modes (so that masses and energies for

perpendicular electric flux will behave as  $1/\tilde{L}$  in weak coupling), there might still be some obstacles to overcome.

To be able to trust the perturbative expansion to give predictions for the long-distance limit, one should at least see the restoration of the cubic symmetry, which at weak coupling is broken when  $\mathbf{m} \neq \mathbf{0}$ , and makes electric fluxes perpendicular to  $\mathbf{m}$  behave very differently to those parallel to  $\mathbf{m}$ . The restoration of this symmetry involves tunneling, but now through a classical potential barrier of order  $g^{-2}$ , and would set in much later. Finally, we cannot escape the intuitive feeling that in the long-distance limit, (approximate) zero momentum modes dominate the dynamics. The analysis for zero magnetic flux has clear advantages: these modes play a role all the way from the shortest distance scales, the nonabelian nature of the theory is essentially incorporated, and the results are directly comparable with existing Monte Carlo data for elongated lattices [10].

## 7. Conclusion

We have given the rigorous short-distance expansion for the energy of electric flux and the mass shifts in the  $0^+$  sector due to tunneling. Thereby we have sorted out many of the problems stated in refs. [1] and [6]. Since tunneling sets in so rapidly and at such a low value of  $z$ , a semiclassical approximation is not good enough. The situation is far more intricate than originally hoped [1,2,6]. We described how to go beyond this semiclassical approximation and we see no essential obstruction to bridging the gap between the short-distance expansion and the Monte Carlo data. We presented circumstantial evidence (e.g. fig. 2), and hereby rest our case, asking for some time to acquire the proof.

Although it is indeed very useful to do Monte Carlo calculations for twisted lattices [17], both for twists in the elongated direction to measure  $\mathcal{E}(z)$ , or in the spatial directions to suppress finite size effects (as also emphasised by the authors of ref. [12]), we would also like to urge strongly for improvements in existing Monte Carlo data in the small  $z$  regime.

For the more distant future one might think of a way in which Monte Carlo calculations can be used to derive an effective theory for the long wavelength modes, in the regime where the weak coupling result for the hamiltonian will be inapplicable. From the analytic side we would of course like to include fermions and to extend the calculations to  $SU(3)$  [31]. There are no fundamental obstacles to doing this.

The numerical calculations were done on the Ridge 32 at the ITP in Stony Brook. We made extensive use of the symbolic manipulation program SMP [21] and the IMSL routines for numerical integration and finding matrix eigenvalues.

It is clear that this paper could never have been written without the seminal work by Gerard 't Hooft and Martin Lüscher; it is therefore a pleasure for one of us

(P.v.B.) to greatly acknowledge their advice. We especially wish to thank Martin Lüscher for providing his personal notes, useful for setting up the numerical calculations, and for comments on the previous version of this paper. Furthermore we thank M. Atiyah, Bernd Berg, Michael Creutz, Hans Hansson, J. Jurkiewicz, Chris Korthals Altes, Daniel Rohrlich, Herman Verlinde and Peter Weisz for discussions. We also acknowledge Bernd Berg for informing us on the latest status of the mass gap. This work was supported in part by NSF grant numbers Phy-85-07627 and DMS-84-05661.

### Appendix A

This appendix discusses the consistency of the tunneling approach as alluded to in ref. [2] and the discussion below eq. (18). The consistency is summarised by demanding that the following diagram commute.

$$\begin{array}{ccc}
 \mathcal{L} = -\frac{1}{2g_0^2} \text{Tr}(F_{\mu\nu}^2) & \xrightarrow{A_\mu^a \setminus C_i} & \mathcal{L}_{\text{eff}} = \frac{LC^2}{2g_R^2} - V_1(C) + \dots \\
 & \searrow^{A_\mu^a \setminus c_i^a} & \nearrow^{c_i^a \setminus C_i} \\
 & \mathcal{L}_{\text{eff}} = \frac{L}{2g_R^2} \text{Tr}(F_{\mu\nu}^2(c)) + \dots &
 \end{array}$$

An arrow with  $A \setminus B$  means that one is obtained from the other by integrating out all fields  $A$  except for  $B$ , thus yielding an effective lagrangian in  $B$ . From a hamiltonian point of view this was described by Lüscher [4], using Bloch degenerate perturbation theory, and from the lagrangian point of view, using a suitable nonlocal gauge fixing, by one of us [2b]. It was observed in the main text that the potential  $V_1(C)$  had to coincide with the effective potential derived in eq. (18) in the region where both the one-loop effective potential along the vacuum valley and Lüscher's perturbative hamiltonian could be trusted, i.e.  $g^{2/3} \ll |C| \ll 1$ . We verified this to lowest order by comparing eq. (19a) and eq. (19b). We can push this to higher order, as was indicated in ref. [2b].

Lüscher's effective hamiltonian is given by [4, 6]:

$$\begin{aligned}
 LH' = & \frac{g^2}{2} \frac{\partial^2}{\partial c_i^{a2}} + \frac{1}{4g^2} (c_k^a c_k^a c_l^b c_l^b - c_k^a c_k^b c_l^a c_l^b) \\
 & + \kappa_1 c_k^a c_k^a + \kappa_3 (c_k^a c_k^a c_l^b c_l^b + 2c_k^a c_l^a c_k^b c_l^b) \\
 & + \kappa_4 (5c_k^a c_k^a c_k^b c_k^b - c_k^a c_k^a c_l^b c_l^b - 2c_k^a c_l^a c_k^b c_l^b) + \dots \quad (A.1)
 \end{aligned}$$

(We ignore a term which can be absorbed by a finite rescaling of the coupling constant and the fields, i.e. the term proportional to  $\kappa_2$  in ref. [6], eq. (10e).) The numerical values for  $\kappa_i$  are [4, 6]:

$$\kappa_1 = -0.30104661 \dots, \quad \kappa_3 = \frac{4}{45}(4\pi)^{-2}, \quad \kappa_4 = -0.15687855 \dots \cdot 10^{-2}. \quad (\text{A.2})$$

To lowest order in  $g$ , the corrections to eq. (19a) due to the terms proportional to  $\kappa_1$ ,  $\kappa_3$ , and  $\kappa_4$  are simply obtained by restricting these terms to the vacuum valley. Hence eq. (19a) gets replaced by:

$$\left( -\frac{g^2}{2C^2} + 2|C| + \kappa_1 C^2 + 3(\kappa_3 - \kappa_4)(C^2)^2 + 5\kappa_4 C_i^4 + \dots \right) / L. \quad (\text{A.3})$$

Consistency requires that all terms independent of  $g$  should coincide with the effective one-loop potential [2, 4]

$$V_1(C) = \frac{4}{\pi^2 L} \sum_{n \neq 0} \frac{\sin^2(n \cdot \frac{1}{2}C)}{(n^2)^2}. \quad (\text{A.4})$$

Hence we make a Taylor expansion of eq. (A.4). Using lattice sum techniques [2a, 22] one easily derives that the singular leading term is  $2|C|/L$  and that

$$W(C) = LV_1(C) - 2|C| \quad (\text{A.5})$$

is analytic for all  $C$ . Using the cubic symmetry this leads to:

$$\begin{aligned} W(C) = & \frac{1}{2} \frac{\partial^2 W(C)}{\partial C_1^2} \Big|_{C=0} C^2 \\ & + \frac{1}{48} \left( \Delta^2 - 3 \frac{\partial^4}{\partial C_1^4} \right) W(C) \Big|_{C=0} \sum_{i \neq j} C_i^2 C_j^2 + \frac{1}{24} \frac{\partial^4 W(C)}{\partial C_1^4} \Big|_{C=0} \sum_i C_i^4 + \dots \end{aligned} \quad (\text{A.6})$$

One easily evaluates  $\Delta^2 W(0)$  to be  $2/\pi^2$ . The remaining terms can all be expressed in terms of the Taylor coefficients for  $W$  along one of the axes. There exists a rapidly converging expression for this function [2] ( $C \in [-2\pi, 2\pi]$ ):

$$W(Ce_1) = -\frac{1}{\pi} C^2 + \frac{8}{\pi} \sum_{n=1}^{\infty} a_n \sin^2\left(\frac{1}{2}nC\right), \quad (\text{A.7})$$

with [2a, 20]:

$$a_n = \frac{2\pi}{n} \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} K_1(2\pi n|k|) < \frac{5\pi^2}{n} e^{-2\pi n}. \quad (\text{A.8})$$

Here  $K_1$  is the modified Bessel function. The expansion of eq. (A.4) now becomes:

$$\begin{aligned} LV_1(C) = & 2|C| + \left( \frac{2}{\pi} \sum_{n=1}^{\infty} n^2 a_n - \frac{1}{\pi} \right) C^2 \\ & + \left( \frac{1}{24\pi^2} + \frac{1}{4\pi^2} \sum_{n=1}^{\infty} n^4 a_n \right) (C^2)^2 - \left( \frac{1}{24\pi^2} + \frac{5}{12\pi} \sum_{n=1}^{\infty} n^4 a_n \right) C_i^4 \dots \end{aligned} \quad (\text{A.9})$$

Using the numerical values of  $a_n$  [2], this indeed coincides with eq. (A.3), and we therefore establish the consistency up to fourth order in  $C$  and one-loop in  $g$ ; the  $O(g^2)$  term in eq. (A.3) is a two-loop contribution [20].

Note that since  $V_1(C)$  in eq. (A.4) is minimal along the axes [2, 20], the minimal potential height the wave function has to penetrate is  $(\pi + (8/\pi) \sum_{n=1}^{\infty} a_{2n+1})/L$ , which roughly equals the ground state energy for  $z = 1.2$ . This is in good agreement with the fact that tunneling sets in at  $z = 1.2$ .

## Appendix B

Here we will derive the short-distance expansion for the heavy quark potential. In the Born approximation this is obtained by computing the connected two-point function. Using the cubic symmetry one has

$$\langle A_i^a(\mathbf{R}) A_j^b(\mathbf{0}) \rangle_C = -\delta^{ab} \delta_{ij} \frac{1}{3} V_L(\mathbf{R}). \quad (\text{B.1})$$

The factor 3 is such that for a rectangular Wilson loop with sides  $|\mathbf{R}|$  and  $T$ ,  $\langle W(|\mathbf{R}|, T) \rangle = \exp(-TV_L(\mathbf{R}))$  for  $T \rightarrow \infty$ . We can next make a momentum expansion:

$$A_i^a(\mathbf{R}) = \sum_{\mathbf{p}=0}^{\infty} \frac{A_i^a(\mathbf{p})}{L^{3/2}} e^{2\pi i \mathbf{p} \cdot \mathbf{R}/L}, \quad (\text{B.2})$$

and evaluate eq. (B.1) to lowest order in the coupling constant. We will not take the tunneling effects into account for this calculation. Hence one can compute the

connected two-point function by expanding around  $A = 0$ , and one finds:

$$\langle A_i^a(\mathbf{R}) A_j^b(\mathbf{0}) \rangle_C = \sum_{\mathbf{p}=\mathbf{0}}^{\infty} \frac{\langle A_i^a(\mathbf{p}) A_j^b(\mathbf{p}) \rangle}{L^3} e^{2\pi i \mathbf{p} \cdot \mathbf{R} / L}. \quad (\text{B.3})$$

To lowest order the hamiltonian for the nonzero momentum modes is given by

$$-\frac{g_0^2}{2} \frac{\partial^2}{\partial A_i^a(\mathbf{p})^2} + \frac{1}{2g_0^2} \left( \frac{2\pi \mathbf{p}^2}{L} \right)^2 A_i^a(\mathbf{p}) A_i^a(\mathbf{p}). \quad (\text{B.4})$$

For the zero momentum modes one can use Lüscher's effective hamiltonian (A.1). Hence we find:

$$\langle A_i^a(\mathbf{R}) A_j^b(\mathbf{0}) \rangle_C = \frac{\langle c_i^a c_j^b \rangle}{L} + \delta_{ij} \delta^{ab} \sum_{\mathbf{p} \neq \mathbf{0}} \frac{g_0^2 e^{2\pi i \mathbf{p} \cdot \mathbf{R} / L}}{16\pi^2 L \mathbf{p}^2}. \quad (\text{B.5})$$

The one-loop corrections will turn the bare coupling constant in eq. (B.5) into a renormalised one, and the vacuum expectation value of  $c_i^a c_i^a$  can easily be obtained from the results in ref. [6] (cf. eq. (A.1))

$$\langle c_i^a c_j^b \rangle = \delta_{ij} \delta^{ab} (g^{4/3} \varepsilon_2 + 2g^2 \varepsilon_3 + O(g^{8/3})) / 9\kappa_1, \quad (\text{B.6})$$

where  $E = g^{2/3} \varepsilon / L = (g^{2/3} \varepsilon_1 + g^{4/3} \varepsilon_2 + g^2 \varepsilon_3 + O(g^{8/3})) / L$  is the ground state energy. Substituting the expression for  $z = M_L(0^+) L$  (eq. (29)), and the values of  $\varepsilon_2$  and  $\varepsilon_3$  (ref. [6]) and  $\kappa_1$  (eq. (A.2)) one obtains eqs. (36)–(38). Using lattice sum techniques [2a, 22] a rapidly converging expression for  $U(\mathbf{r})$  is found:

$$\begin{aligned} U(\mathbf{r}) &= \frac{3}{16\pi^2} \sum_{\mathbf{n} \neq \mathbf{0}} \frac{\cos(2\pi \mathbf{n} \cdot \mathbf{r})}{\mathbf{n}^2} \\ &= \frac{3}{16\pi^2} \left\{ -\pi + \sum_{\mathbf{n} \neq \mathbf{0}} \frac{e^{-\pi \mathbf{n}^2}}{\mathbf{n}^2} e^{2\pi i \mathbf{n} \cdot \mathbf{r}} + \sqrt{\pi} \sum_{\mathbf{n}} \frac{\text{erfc}(\sqrt{\pi} |\mathbf{n} - \mathbf{r}|)}{|\mathbf{n} - \mathbf{r}|} \right\}. \quad (\text{B.7}) \end{aligned}$$

## Appendix C

In this appendix we will derive the expression for the energy of electric flux obtained from twisted boundary conditions. Let  $L^3 \times T$  be the extent of the four-dimensional torus on which twisted boundary conditions are defined [1] and  $\mathbf{k}$  the twist in the time direction. Then one can express the twisted partition function



in terms of the free energies  $F(\mathbf{e}, L, T, \beta)$  introduced by 't Hooft [1]:

$$Z_k = \sum_{\mathbf{e}} e^{-2\pi i \mathbf{k} \cdot \mathbf{e} / N} e^{-F(\mathbf{e}, L, T, \beta)}. \quad (\text{C.1})$$

Here  $\beta = 4/g_0^2$  and  $\mathbf{e}$  is the electric flux, integer mod  $N$  for  $\text{SU}(N)$ . The free energy behaves as

$$F(\mathbf{e}, L, T, \beta) = T\Delta E(\mathbf{e}, L, \beta), \quad (\text{C.2})$$

up to corrections  $\exp(-TM)$ , with  $M$  the gap between the ground state and the first excited state in a given electric sector (assumed to be finite); for zero electric flux  $M$  is by definition the glueball mass. Next we can use the cubic symmetry and normalize such that  $\Delta E(\mathbf{0}, L, \beta) = 0$ . Hence  $\Delta E(\pm \mathbf{e}_i, L, \beta) = \Delta E$  ( $\mathbf{e}_i$  the unit vector in the  $i$ -direction). Finally it is reasonable to assume a finite gap between the energy of one unit of electric flux and two units of electric flux, of order  $\Delta E$ . Then one easily derives:

$$\frac{Z_k}{Z_0} = 1 + 2 \sum_{i=1}^3 (\cos(2\pi k_i / N) - 1) e^{-T\Delta E} + O(e^{-(\Delta E + \delta E)T}), \quad (\text{C.3})$$

which is easily seen to give ( $\delta E = \min(M, \Delta E)$ ):

$$\Delta E = -\frac{1}{T} \ln \left[ \frac{1}{4T} \frac{\partial}{\partial \beta} (\ln(Z_k/Z_0)) \right] / \left( \sum_{i=1}^3 \sin^2(\pi k_i / N) \frac{\partial \Delta E}{\partial \beta} \right) + O(T^{-1} e^{-T\delta E}). \quad (\text{C.4})$$

In lattice units  $L = N_s$ ,  $T = N_t$  and thus for  $\mathbf{k} = (1, 0, 0)$  and  $\text{SU}(2)$ , with

$$\mu(\beta, N_t) = -\frac{1}{2} \frac{\partial}{\partial \beta} \ln(Z_k/Z_0), \quad (\text{C.5})$$

we find eq. (42). For  $N_t \rightarrow \infty$  one can ignore the term  $[2] - 2 \partial \Delta E / \partial \beta$ , but for finite  $N_t$  it can give an appreciable correction.

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