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ON THE RATIO OF THE STRING TENSION AND THE GLUEBALL MASS SQUARED IN THE CONTINUUM

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We present the weak coupling non-perturbative expression for the energy of 't Hooft type electric flux. Combining this with Lüscher's perturbative scheme for the glueball mass $M(0^+)$, we propose a method to estimate the ratio $\sigma/M(0^+)^2$, with σ the string tension.

1. Introduction

It is believed that pure gluon dynamics is responsible for confinement through the formation of electric flux tubes. This is a long-distance phenomenon, which is plagued by infrared divergences and the strongness of the coupling constant. Nevertheless, these features are considered crucial for the confinement problem, but they make calculations practically impossible. A way to avoid both problems at the same time is to formulate the Yang-Mills gauge theory in a finite volume. This provides an infrared cut-off and guarantees a small coupling constant for a sufficiently small volume. By choosing the appropriate boundary conditions one can hope that a connection can be made between the short- and long-distance behaviour.

By formulating pure SU(2) gauge theories on a cube of size $L \times L \times L$ one can use twisted boundary conditions [1] to introduce the gauge invariant notion of electric flux. Hence this paper will describe results for SU(2)/Z₂ Yang-Mills on the torus T³ [2]. The centre Z₂ of the gauge group SU(2) seems to play an important role; for one thing it does not allow dynamical quarks in the fundamental representation. This is similar to breaking the string by production of a quark-anti-quark pair. The physical interpretation of twisted boundary condition is, that they simulate static quark sources, "spread" out over the sides of the cube. These boundary conditions are especially well suited to calculate the energy of electric flux. Although, at this stage, we will not be able to prove electric flux tube formation; assuming formation of strings for large-L implies [1] that ΔE behaves as $\sigma \cdot L$ with σ the string tension and ΔE the energy of 1 unit of electric flux.

 Z_N vortices have been proposed to describe the dynamics for flux tube formation [3]. Recently this mechanism has been questioned, based on a relation between the

fundamental string tension and a (scaling) linear piece in the potential for other representations [4]. Most dramatically for $N \rightarrow \infty$, where confinement seems to survive (for a review, see [5]), one has an adjoint potential twice the one for the fundamental representation, if factorization is rigorously true [6]. But the adjoint representation is invariant under a Z_N transformation. Apart from this, there are other reasons why one suspects that the centre of the gauge group does not play a *direct* dynamical role [7]. Our method of calculation is independent of the specific dynamics for confinement, but might give a clear hint towards a better understanding, if the short-to-long distance connection leads to an acceptable result.

This paper will concentrate on the general principles. Sect. 2 describes SU(2) gauge fields on the torus and the definition of 't Hooft type electric flux. In sect. 3 the calculation of the energy of electric flux is exhibited, by pointing out the similarity with certain toy models [8]. We also explain how Lüscher's perturbative results [9] can be obtained from a lagrangian point of view, introducing a new type of gauge fixing. Sect. 4 gives the results for the energy of electric flux in weak coupling by expressing $\Delta E/M_L(0^+)$ as a function of the universal expansion parameter [10] $z = M_L(0^+)L$, where $M_L(0^+)$ is the glueball mass in a finite volume L^3 . In this way we obtain a renormalization group independent function, whose behaviour for large z is also known (supposing flux tube formation). We propose a "minimal" short-tolong distance connection which gives a way of estimating $\sigma/M(0^+)^2$. In sect. 5 we discuss the results.

Technical details, generalization to SU(N) and a final numerical determination of one of the constants will be published soon.

2. SU(2) Yang-Mills on a torus

The space we work on is the 3-dimensional torus, specified by a cube $L \times L \times L$. In the absence of magnetic flux the gauge potentials can be chosen periodic [9] and Gauss's law specifies the remaining gauge freedom (e_i the unit vector in the *i*th direction):

$$\Omega(\mathbf{x} + L\mathbf{e}_i) = (-1)^{k_i} \Omega(\mathbf{x}) . \tag{1}$$

k (integer mod 2) is a topological invariant (the twist n_{i4} [1]). States in the physical Hilbert space are labelled by the representations of the homotopy group. The variable conjugate to k is e (integer mod 2) and defines the electric flux [1, 11], in complete analogy with the instanton θ -parameter [12] (also present on T³, but ignored here).

For weak coupling we consider the minimum of the classical potential given by $(1/4g^2) \int_{T_3} \text{Tr} (F_{ij}^2)$, with $F_{\mu\nu}$ the Yang-Mills curvature. For SU(2), up to a periodic gauge, this minimum is at [9]

$$A = c\sigma_3/2L$$
, c spatially constant, (2)

where c and $c+2\pi k$ are gauge equivalent with a gauge function of the type (1). Hence the potential exhibits a vacuum valley. One might expect that the ground state wave function(al) spreads out along this valley. However the shape of the potential perpendicular to the vacuum valley depends on c. Perturbative wave function(al)s will be concentrated around points where the potential is widest [9]. This can be thought of to be due to the zero-point fluctuations in the perpendicular direction depending on c, hence inducing an effective c-dependent potential. The one-loop approximation for this potential was calculated by Lüscher [9]:

$$V_1(c) = \frac{4}{L} \sum_{n \neq 0} \frac{\sin^2(n \cdot c)}{\pi^2(n^2)^2}.$$
 (3)

It has the following properties [13] ($c = 2\pi k + d$):

$$V_{1}(c) = 2L^{-1}|d|, \qquad |d| \ll 1, \qquad (4)$$

$$V_{1}(c) \equiv V_{1}(ce_{1}) = \frac{|d|(2\pi - |d|)}{L\pi} + \frac{8}{L\pi} \sum_{n=1}^{\infty} a_{n} \sin^{2}(\frac{1}{2}nc), \qquad (4)$$

$$(d \in [-2\pi, 2\pi], \qquad a_{1} = 2.7052746 \times 10^{-2}, \qquad a_{2} = 1.6048745 \times 10^{-5}, \qquad a_{3} = 1.6065690 \times 10^{-8}, \qquad a_{4} = 1.9385627 \times 10^{-11}, \ldots). \qquad (5)$$

Perturbative wave function(al)s are therefore concentrated around $c = 2\pi k$ and there is a degeneracy in e for all orders in perturbaton theory [9]. The energy split between states with different electric flux e is thus caused by tunneling through a potential barrier, induced by quantum corrections and will only be suppressed by a factor $\exp(-S/g)$ instead of the familiar factor $\exp(-S/g^2)$. Vacuum valleys imply zeroaction instanton solutions (the so-called twist-eating solutions [14]), which dominate over the non-zero action instantons [15]. Because their action is quantum induced by the squeezing of the potential in-between two perturbative vacuum states we propose to call these configurations *pinchons*.

3. (Non-)perturbative calculations

A complication in the calculation is caused by the fact that at $c = 2\pi k$, the quadratic approximation in the perpendicular direction of the potential breaks down. Explicitly for k = 0, the potential is quartic in the spatially constant off diagonal modes.

The 1-loop approximation therefore breaks down at $c = 2\pi k$. For perturbation theory, expanding around A = 0, Lüscher therefore integrated out the non-constant modes, to be left with an effective hamiltonian in the constant (including off-diagonal) modes. In weak coupling the 1-loop approximation is sufficient to determine the low-lying energies.

In the lagrangian formalism this can be reproduced by choosing the following non-local gauge fixing (P the projector $L^{-3} \int_{T^3}$):

$$\chi = (1 - P)(\partial_{\mu}A_{\mu} + i[PA_{\mu}, A_{\mu}]) + PA_{0}.$$
(6)

The effective lagrangian in 1-loop can be obtained in a background type of calculation. [But except for the finite part in the renormalization of the coupling constant the 1-loop result can be uniquely obtained from a Taylor expansion of V_1 in eq. (3), giving the following expressions for the parameters κ_1 , κ_3 and κ_4 as introduced in ref. [16]:

$$\kappa_{1} = -\left(1 - 2\sum_{n=1}^{\infty} n^{2} a_{n}\right) / \pi, \qquad \kappa_{3} = (180\pi^{2})^{-1},$$

$$\kappa_{4} = -\left(1 + 10\pi\sum_{n=1}^{\infty} n^{4} a_{n}\right) / 120\pi^{2}.$$
(7)

Substituting the values of a_n reproduces the numerical values of ref. [16] for κ_1 and κ_4 to 8 significant digits!] In lowest order the effective lagrangian is simply (such that $S = \int dt L_1$):

$$L_{1} = \frac{L^{3}}{4g_{\rm R}^{2}} \operatorname{Tr} \left(F_{\mu\nu}^{2} \right), \qquad (8)$$

with A_{μ} spatially constant and in the axial gauge $A_0 = P\chi = 0$. The energy of the ground state behaves as $E = L^{-1}g_R^{2/3}(\varepsilon + O(g_R^{2/3}))$; g_R is the renormalized coupling constant.

There are some reasons why (6) is not suitable for non-perturbative calculations and one is led to a gauge fixing which allows us to compute an effective lagrangian in c with the same cubic symmetry as V_1 . This would reduce the problem to a crystal type calculation for the low-lying energy levels, with the lattice momenta restricted to e. The appropriate non-local gauge fixing is

$$\chi_3 = (1 - P_3)(\partial_\mu A_\mu + [P_3 A_\mu, A_\mu]) + P_3 A_0, \qquad (9)$$

with P_3 the projection on the constant diagonal modes:

$$P_3 A_\mu = \sigma_3 \int_{\mathrm{T}^3} \mathrm{Tr} \, (A_\mu \sigma_3) / 2L^3$$

One can easily verify that for $|d| \ge g^{2/3}$ (with $c = 2\pi k + d$), up to the relevant order, the 1-loop approximation is given by [13]

$$L_{1} = \frac{L\dot{c}^{2}}{2g_{R}^{2}} - V_{1}(c) .$$
 (10)

In the neighborhood of $c = 2\pi k$ this approximation breaks down. However the full effective lagrangian has the cubic symmetry of $V_1(c)$ because χ_3 is invariant under the gauge transformations which map c into $c + 2\pi k$. A word of caution: There will be ghost zero modes, related to gauge transformations which leave A_{μ} invariant; they should *not* be confused with Gribov ambiguities [17].

In a hamiltonian formulation one can define Lüscher's effective hamiltonian [9] in the spatially constant potentials with c in the unit cell $|c_i| \leq \pi$. For the other cells

 $(|c_i - 2\pi k_i| \le \pi)$ one derives the same effective hamiltonian, but as a function of appropriately gauge transformed fields. As will be explained in a subsequent paper, suitable identification will lead to a (in c) periodic hamiltonian with a finite number of degrees of freedom. To the *relevant* order in the coupling constant and fields, required for the tunneling calculation

$$H_{\text{eff}} = \frac{-g_{\text{R}}^2}{2L} \frac{\partial^2}{(\partial z_i^a)^2} - \frac{1}{2g_{\text{R}}^2 L} \operatorname{Tr} [z_i, z_j]^2 - \frac{2}{L} |c| + V_1(c), \qquad |c_i| \le \pi, \qquad (11)$$

with g_R the renormalized coupling constant and

$$\boldsymbol{A} = \frac{\boldsymbol{z}}{L} = \frac{\boldsymbol{z}^a}{2L} \,\boldsymbol{\sigma}_a \,. \tag{12}$$

The vacuum valley is labelled by $z^a = cr^a$ (*r* is a fixed unit vector). From this hamiltonian a one-loop effective lagrangian in the variable *c* leads to the same result as eq. (10), for $g_R^{2/3} \ll |c_i| \ll \pi$.

The path decomposition expansion of ref. [18] tells us how to match perturbative and tunneling contributions for a situation where ordinary instanton techniques break down. The perturbative contribution is just the lowest (non-trivial) order ground state wave function and energy. The latter was already determined numerically by Lüscher and Münster [16]; for the wave function identical techniques will be used. Here we will "derive" the expression for ΔE by referring to toy models which exhibit the main features of our present problem. This will fix ΔE up to an overall constant. The simplest toy model describing the features of a vacuum valley and quartic mode problems is given by the hamiltonian

$$H = -\frac{g^2}{2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{2g^2} (x^2 - 1)^2 y^2.$$
(13)

From the detailed analysis of ref. [8] it follows that ΔE is determined up to a constant by the 1-loop approximated potential (for eq. (13): $V_1(x) = \frac{1}{2}|x^2 - 1|$) provided one uses the true perturbative ground state energy. By analogy, eq. (10) can be used to calculate the energy of electric flux up to an overall constant. Since $V_1(c)$ behaves as |c| for c close to zero we find

$$\Delta E = \tilde{A} g_{R}^{5/3} \exp\left(-g_{R}^{-1} \int_{0}^{2\pi} \sqrt{2L(V_{1}(c) - E_{0})} \, \mathrm{d}c\right).$$
(14)

The $g_R^{5/3}$ can be inferred from the expression of ΔE for the double-cone model [8]

$$H = -\frac{1g^2}{2}\frac{\partial^2}{\partial x^2} + \sqrt{(|x_1| - \pi)^2 + x_2^2 + x_3^2}$$
(15)

and E_0 is the true perturbative gound state energy. Furthermore one can verify that the path of minimum action connecting two nearest-neighbor minima of $V_1(c)$ is a straight line.

In the approximation $a_n \equiv 0$ (see eq. (5)) one finds $S = \int_0^{2\pi} \sqrt{2LV_1(c)} dc \simeq \frac{1}{2}\pi^2 \sqrt{2\pi}$ and $T = \int_0^{2\pi} (1/\sqrt{2LV_1(c)}) dc \simeq \frac{1}{2}\pi \sqrt{2\pi}$. We will call the instanton solution for the lagrangian of eq. (10) the pinchon solution. In the approximation $a_n = 0$ this pinchon solution becomes

$$c(t) = 2\pi e^{(1)} \sin^2(\pi t g_{\rm R}/TL).$$
(16)

4. The short-to-long distance connection

We now present our results. In weak coupling for SU(2), ΔE is given by (see eq. (14); an explicit factor L^{-1} appears for dimensional reasons):

$$\Delta E = AL^{-1}g(L)^{5/3} \exp\left(-Sg(L)^{-1} + T\varepsilon g(L)^{-1/3}\right)\left(1 + O(g(L)^{\delta})\right), \qquad (17)$$

with $\delta \ge \frac{1}{6}$ and

$$S = 12.4637 \cdots, \qquad T = 3.9186 \cdots, \qquad \varepsilon = 4.11672 \cdots.$$
 (18)

The calculation of the prefactor A is presently under investigation and is expected to be significantly smaller than 1 [8]. g(L) is the renormalized coupling constant at the scale L. To lowest order we have

$$g(L) = \left[-\frac{11}{12\pi^2} \ln \left(\Lambda L \right) \right]^{-1/2}.$$
 (19)

One is sadly rather ignorant about the large-L behaviour of this equation. This would make a fit to the strong coupling domain virtualty impossible. However, g(L) can be expressed in Lüscher's universal expansion parameter $z = M_L(0^+)L$ [10], where $M_L(0^+)$ is the glueball mass for a finite volume. Expansions in z are renormalization group independent. One can include in z all higher-order corrections to eq. (19) (even including in principle non-perturbative corrections). Expressing eq. (17) in terms of z, should hence give a result relatively insensitive to these corrections. For the case of SU(2) gauge fields on T³, z was determined by Lüscher and Münster [16]:

$$z \equiv M_L(0^+)L = c_1 g(L)^{2/3} + c_2 g(L)^{4/3} + O(g(L)^2), \qquad (20)$$

with

$$c_1 = 2.269 \cdots, \qquad c_2 = -0.7974 \cdots.$$
 (21)

To eliminate all L and renormalization group dependence we consider $\Delta E/M_L(0^+)$ and find

$$\Delta E/M_L(0^+) = Ac_1^{-1}\hat{z}^{3/2} \exp\left(-S\hat{z}^{-3/2} + B\hat{z}^{-1/2}\right)(1 + O(\hat{z}^{3\delta/2})), \qquad (22)$$

with

$$B = T\varepsilon - 3c_2 S/2c_1 = 22.701 \cdots, \qquad \hat{z} = z/c_1.$$
(23)

Fig. 1 gives the graph for this function. We see that tunneling sets in around z = 2 and there is a remarkable linear behaviour between z = 2.5 and z = 3.5. Considering the previous arguments it is tempting to believe that eq. (22) is accurate up to $z \sim 3.5$, with the only error determined by the perturbatively calculable $O(\hat{z}^{3\delta/2})$ term.

For large z eq. (22) should behave as

$$\Delta E / M_L(0^+) = (z + \varepsilon_0) (\sigma / M(0^+)^2) + O(1/z) , \qquad (24)$$

where we assumed formation of strings [1] and that $M_L(0^+)$ tends exponentially [19] to the infinite-volume glueball mass $M(0^+)$. We therefore match eq. (22) to eq. (24) at $z_1 = 2.9195 \cdots$, where the second derivative of $\Delta E/M_L(0^+)$ (eq. (22)) is identical zero. This yields for $z > z_1$:

$$\Delta E/M_L(0^+) \simeq 9.1 \times 10^4 (z - 2.24 \cdots) A, \qquad (25)$$

hence $\sigma/M(0^+)^2 \simeq 9.1 \times 10^4 A$ which, compared to the value [20] of $0.1 \sim 0.2$ obtained by Monte Carlo calculations^{*} means that A should be of the order 10^{-6} , which is quite feasible. Note that ΔE contains a negative constant contribution which is not present in the static $q - \bar{q}$ potential [22]. This is of course due to the fact that the charge in our case is "spread out".



Fig. 1. The graph for the short-distance behaviour of $\Delta E/M_L(0^+)$ (eq. (22)) in units of 10⁵A, matched to the long-distance linear ansatz (eq. (24)).

* Recent glueball estimates favor $\sigma/M(0^+)^2 \sim 0.1$ [21].

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5. Discussion

We studied in this paper a method for calculating energy of electric flux at weak coupling (small L), whose non-zero value is caused by non-perturbative effects. In going from small to large distances a cross-over in ΔE and $M_L(0^+)$ will occur, supposedly at the same value of L. Studying $\Delta E/M_L(0^+)$ as a function of the universal expansion parameter $z = M_L(0^+)L$ gives two reasons to believe that in this function no cross-over will appear. First it might cancel out in the ratio $\Delta E/M_L(0^+)$, second (if present) it will be smoothed due to the cross-over in z(L) (a small range in L around the cross-over corresponds to a large range in z).

Since we have no control over the reliability of our "minimal" scheme for the short-to-long distance connection, the value of $\sigma/M(0^+)^2$ cannot really be considered as a prediction. We can however compare our results with Monte Carlo data (even for SU(3) we have to resort to these since a glueball has not been seen yet, and dynamical quarks are not incorporated in the calculations). If our estimate will roughly correspond with these Monte Carlo data the pinchon must somehow be relevant for flux tube formation. It can however only be a first step in our understanding since the pinchon corresponds to spatially homogeneous excitations. It is also interesting to point out that its dual in the sense defined by 't Hooft [1] is a magnetic domain wall, with a possible implication for the vacuum structure of QCD.

The reason why the expression for ΔE is also interesting in its own right, is that it can (more directly) be compared to numerical simulations. So far only data for square 4-dimensional lattices^{*} are available [14, 23]. For small coupling where the correlation length becomes larger than the linear dimensions of the lattice ($z \le 1$) one however begins to see finite-temperature effects. Since our calculation is for zero temperature, the extension of the lattice in the time direction has to be kept considerably larger than the correlation length. For $z \rightarrow 0$ this temporal extension has to grow as 1/z but for intermediate values of z (not too weak a coupling for a given spatial extension) Monte Carlo calculations should still be feasible. In lattice units ΔE can be determined by [1]

$$\Delta E_{\ell} = \lim_{N_{t} \to \infty} -\frac{1}{N_{t}} \ln \left[-\frac{1}{2N_{t}} \frac{\partial}{\partial \beta} \ln \left(Z_{t} / Z_{u} \right) \right], \qquad (26)$$

where $Z_{t(u)}$ is the twisted (untwisted) partition function [23] for a lattice of size $N_s^3 \times N_t$ at coupling $\beta = 1/g^2$. If at the same time the glueball mass in lattice units $(M(0^+)_\ell)$ is available (for the same β and N_s), one has simply

$$\Delta E/M_L(0^+) = \Delta E_{\ell}/M(0^+)_{\ell}, \qquad z = M(0^+)_{\ell} \cdot N_s.$$
(27)

Feasibility of these numerical simulations will probably lie just within the present day computational limit.

^{*} That is with twisted boundary conditions.

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