Finite size techniques and the strong coupling constant

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LECTURE 3

- **Chiral symmetry**
- **Symanzik O(a) improved action**
- **Improved operators**
- **Non-perturbative renormalization**

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IS QCD THE CORRECT THEORY OF STRONG INTERACTIONS?

IF YES - GREAT ACHIEVEMENT!

Perturbative calculations at HE consistent at $\sim 10\%$ level

For low energy physics e.g. spectrum, phase shifts, matrix elements e.g. F_π...

need NON-PERTURBATIVE METHODS $\rightarrow$ LATTICE

DIFFICULTIES	"SOLUTIONS"
Control cut-off effects	Non-pert. O(a) improvement
Relate lattice data to quantities in the continuum	Non-pert. renormalization, finite-size methods

Lattice QCD (Wilson 1974)

$$\text{Lattice action} : S = S_{\text{gluon}} + S_{\text{quark}}$$

$$S_{\text{quark}} = a^4 \sum_x \overline{\psi}(x)(D + \underset{\substack{\uparrow \\ \text{bare quark mass}}}{m_0})\psi(x)$$

Wilson-Dirac operator:

$$D = \frac{1}{2}\{\gamma_\mu(\underset{\uparrow}{\nabla^*_\mu} + \underset{\uparrow}{\nabla_\mu}) - a\nabla^*_\mu\nabla_\mu\}$$

gauge covariant forward & backward

difference operators

Many good properities but the "Wilson term", $-\frac{1}{2}a\nabla^*_\mu\nabla_\mu$, violates chiral symmetry

Implications of chiral symmetry violation

* chiral Ward identities violated by terms of $O(a)$
* quark mass requires additive renormalization
* the axial current must be renormalized by a finite multiplicative renormalization constant

$\Rightarrow$ not a fundamental problem, but:

- continuum limit is reached only very slowly
- renormalization is more complicated

Symanzik's local effective theory

At energies well below the cutoff, the lattice theory is equivalent to an effective continuum theory

Effective action: $S_{\rm eff} = S_0 + aS_1 + a^2 S_2 + \ldots$

Leading correction: $S_1 = \int \mathrm{d}^4 x \, \mathcal{L}_1(x)$

$\mathcal{L}_1$: linear combination of local fields of dimension 5:

$$
\begin{aligned}
\mathcal{O}_1 &= \overline{\psi}\,\sigma_{\mu\nu}F_{\mu\nu}\psi, \\
\mathcal{O}_2 &= \overline{\psi}\,D_\mu D_\mu\psi + \overline{\psi}\,\overleftarrow{D}_\mu\overleftarrow{D}_\mu\psi, \\
\mathcal{O}_3 &= m\,\mathrm{tr}\left\{F_{\mu\nu}F_{\mu\nu}\right\}, \\
\mathcal{O}_4 &= m\left\{\overline{\psi}\,\gamma_\mu D_\mu\psi - \overline{\psi}\,\overleftarrow{D}_\mu\gamma_\mu\psi\right\}, \\
\mathcal{O}_5 &= m^2\,\overline{\psi}\psi
\end{aligned}
$$

Renormalized local lattice fields are represented by

$$\phi_{\text{eff}} = \phi_0 + a\phi_1 + a^2\phi_2 + \dots$$

(ϕ_i: local operators in the continuum theory)

in sense that renormalized lattice n-pt. correlation fns.

$$G_n(x_1, \dots, x_n) = (Z_\phi)^n \langle \phi(x_1) \dots \phi(x_n) \rangle_{\text{con}}$$

at fixed **separated** points x_k are given by

$$\begin{aligned} G_n(x_1, \dots, x_n) &= \langle \phi_0(x_1) \dots \phi_0(x_n) \rangle_{\text{con}} \\ &-a \int \mathrm{d}^4 y \, \langle \phi_0(x_1) \dots \phi_0(x_n) \mathcal{L}_1(y) \rangle_{\text{con}} \\ &+a \sum_{k=1}^{n} \langle \phi_0(x_1) \dots \phi_1(x_k) \dots \phi_0(x_n) \rangle_{\text{con}} + \mathrm{O}(a^2) \end{aligned}$$

(expectation values on rhs taken in the continuum theory)

Proofs in framework of PT (Symanzik 1982)
– but structural property assumed non-perturbatively

Removal of "redundant operators"

The field equation implies e.g.

$$\mathcal{O}_4 + c\,\mathcal{O}_5 = 0, \qquad c = 2 + \mathrm{O}(g^2)$$

(true when inserted in a correlation function, up to contact terms, which can be absorbed by a redefinition of ϕ_1)

$\Rightarrow$ $\mathcal{O}_4$ may be eliminated in favour of $\mathcal{O}_5$ in correlation functions (with fields at non-zero distances from each other) from which all on-shell can be extracted

$\mathcal{O}_2$ can also be eliminated

$\Rightarrow$ $\mathcal{L}_1 \sim$ linear combination of $\mathcal{O}_1, \mathcal{O}_3, \mathcal{O}_5$

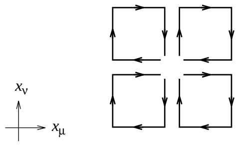
$\mathcal{L}_1$ can be made to vanish by adding O(a) counterterms to the lattice action (Symanzik 1983):

$$\delta S = a^5 \sum_x \left\{ c_1 \widehat{\mathcal{O}}_1(x) + c_3 \widehat{\mathcal{O}}_3(x) + c_5 \widehat{\mathcal{O}}_5(x) \right\}$$

$\widehat{\mathcal{O}}_3$ and $\widehat{\mathcal{O}}_5$ may be dropped, because they correspond to a renormalization of the bare coupling and mass

Improved action (Sheikholeslami & Wohlert 1985)

$$S_{\rm impr} = S_{\rm Wilson} + a^5 \sum_x c_{\rm sw}\, \overline{\psi}(x) \tfrac{i}{4} \sigma_{\mu\nu} \widehat{F}_{\mu\nu}(x) \psi(x)$$

$\widehat{F}_{\mu\nu}(x) \quad \propto \quad \sum$ 

$c_{\rm sw}(g_0)$ tuned to achieve cancellation of O(a) effects

To one-loop order PT:

$$c_{\rm sw} = 1 + 0.2659 \times g_0^2 + \mathrm{O}(g_0^4) \qquad (N = 3,\ \text{any } N_{\rm f})$$

(Wohlert 1987, Lüscher & Weisz 1996)

O(a) improved operators

Bare isovector axial current and density:

$$
\begin{aligned}
A_\mu^a(x) &= \overline{\psi}(x)\gamma_\mu\gamma_5\tfrac{1}{2}\tau^a\psi(x) \\
P^a(x) &= \overline{\psi}(x)\gamma_5\tfrac{1}{2}\tau^a\psi(x)
\end{aligned}
$$

Omitting renormalization factors the improved fields are

$$
\begin{aligned}
(A_{\rm I})_\mu^a &= A_\mu^a + c_{\rm A} a\partial_\mu P^a \\
(P_{\rm I})^a &= P^a
\end{aligned}
$$

$c_{\rm A}$ must be chosen so as to cancel any remaining terms of O(a) in correlation functions involving the axial current

To one-loop order PT:

$$
c_{\rm A} = -0.00756 \times g_0^2 + {\rm O}(g_0^4) \qquad (N=3,\ \text{any } N_{\rm f})
$$

(Lüscher & Weisz 1996)

Mass-independent renormalization schemes

For compatibility with O(a) improvement must have form:

$$\begin{aligned} g_{\rm R}^2 &= \tilde{g}_0^2 Z_{\rm g}(\tilde{g}_0^2, a\mu) \\ m_{\rm R} &= \widetilde{m}_{\rm q} Z_{\rm m}(\tilde{g}_0^2, a\mu) \end{aligned}$$

where the modified bare coupling and mass are defined by

$$\begin{aligned} \tilde{g}_0^2 &= g_0^2\,(1 + b_{\rm g} a m_{\rm q})\,, \quad m_{\rm q} = m_0 - m_{\rm c}, \\ \widetilde{m}_{\rm q} &= m_{\rm q}\,(1 + b_{\rm m} a m_{\rm q}) \end{aligned}$$

Example: the "pole mass" of a free quark is

$$m_{\rm p} = \frac{1}{a}\ln(1 + a m_0) = m_{\rm q} - \tfrac{1}{2} a m_{\rm q}^2 + \ldots$$

$\Rightarrow$ to tree level $b_{\rm m} = -1/2$

Field renormalizations also involve extra factors $1 + b_\phi a m_{\rm q}$

PCAC relation on the lattice

$$\langle \partial_\mu (A_{\rm R})^a_\mu(x)\, \mathcal{O}\rangle = 2m_{\rm R}\langle (P_{\rm R})^a(x)\, \mathcal{O}\rangle + {\rm O}(a^p)$$

$$\begin{aligned} (A_{\rm R})^a_\mu &= Z_{\rm A}(1+b_{\rm A}am_{\rm q})\{A^a_\mu + c_{\rm A}a\partial_\mu P^a\} \\ (P_{\rm R})^a &= Z_{\rm P}(1+b_{\rm P}am_{\rm q})P^a \end{aligned}$$

Define an "unrenormalized current quark mass" through

$$m = \frac{\langle \partial_\mu \{A^a_\mu + c_{\rm A}a\partial_\mu P^a\}\mathcal{O}\rangle}{2\langle P^a\, \mathcal{O}\rangle}$$

The PCAC relation implies

$$m = \frac{Z_{\rm P}(1+b_{\rm P}am_{\rm q})}{Z_{\rm A}(1+b_{\rm A}am_{\rm q})}\, m_{\rm R} + {\rm O}(a^p)$$

$\Rightarrow$ the violation of chiral symmetry can be studied by computing m for various $\mathcal{O}$

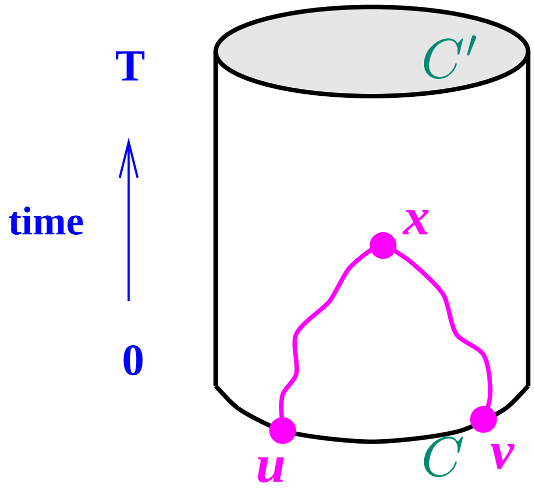
Sources using the Schroedinger functional

Differentiation w.r.t. $\rho, \bar{\rho}$ produces source at bdy. e.g.

$$\mathcal{O}^a = a^6 \sum_{\mathbf{u},\mathbf{v}} \overline{\psi}(u)\gamma_5 \tfrac{1}{2}\tau^a \psi(v), \qquad u_0 = v_0 = 0$$

Correlation functions:

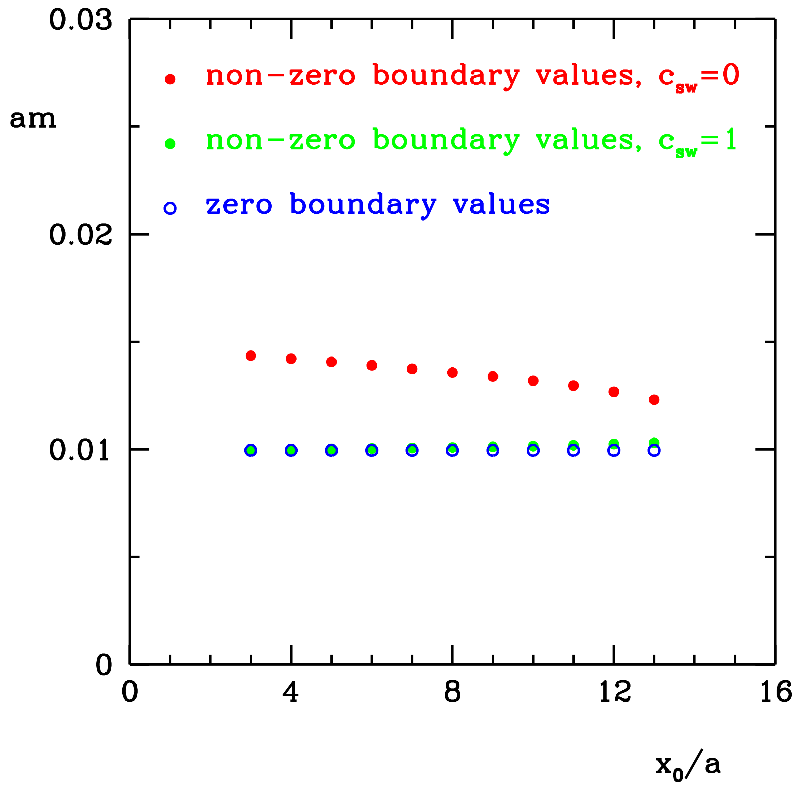
$$\langle A_\mu^a(x)\mathcal{O}^a\rangle = \frac{1}{\mathcal{Z}} \int_{\text{fields}} A_\mu^a(x)\mathcal{O}^a\, \mathrm{e}^{-S}$$



$\Rightarrow$ m must be independent of C, C', x_0 and the lattice sizes T and L, up to cutoff effects

Test of the PCAC relation at tree-level

16×8^3 lattice, $am_0 = 0.01$



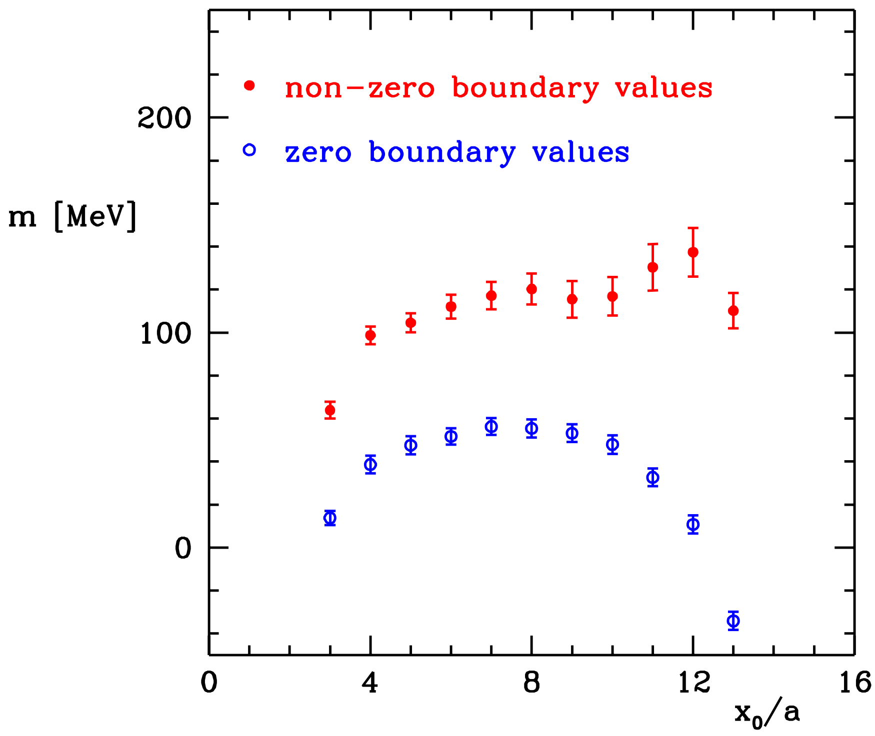
Non-zero boundary values C and C' induce a background gauge field

$\Rightarrow$ test is sensitive to the SW improvement term

$$a^5 \sum_x \tfrac{i}{4} c_{\mathrm{sw}} \overline{\psi} \sigma_{\mu\nu} \widehat{F}_{\mu\nu} \psi, \qquad c_{\mathrm{sw}} = 1 + \mathrm{O}(g_0^2)$$

Unimproved quenched QCD

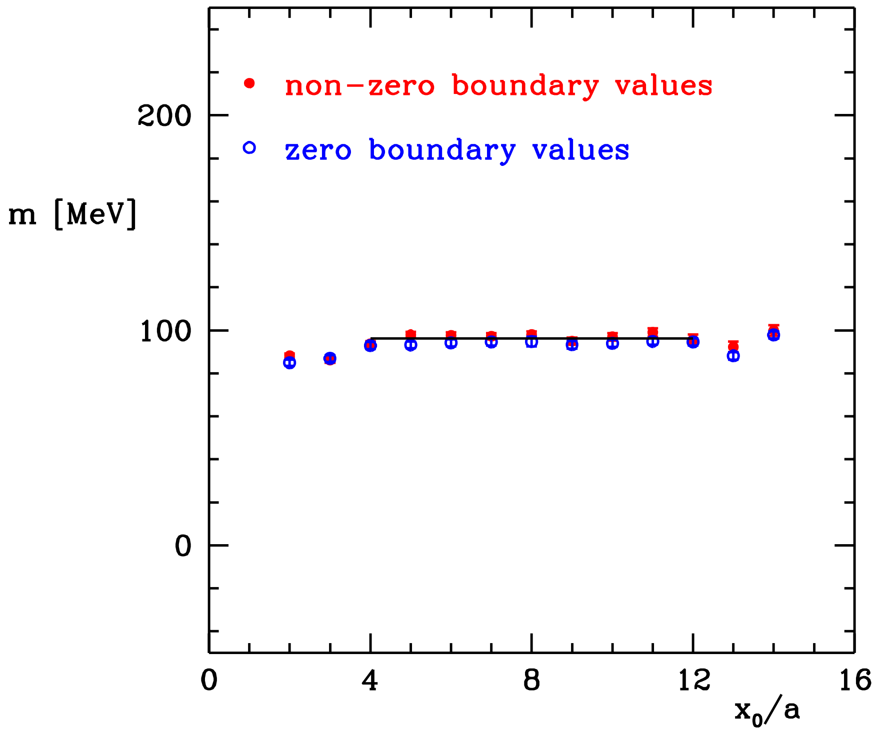
16×8^3 lattice, $\beta = 6.4$, $a = 0.05$ fm



$\Rightarrow$ chiral symmetry is strongly violated

Improved quenched QCD

Same lattice as before, but $c_{\rm sw} = 1.60$, $c_{\rm A} = -0.027$



Similar results are obtained for larger lattices, other boundary values, etc

$\Rightarrow$ at lattice spacings $a \leq 0.1$ fm, and with properly chosen $c_{\rm sw}$ and $c_{\rm A}$, chiral symmetry is effectively restored

Non-perturbative determination of c_{sw} and c_{A}

Take three configurations of $\{x_0, C, C', \ldots\}$ and require

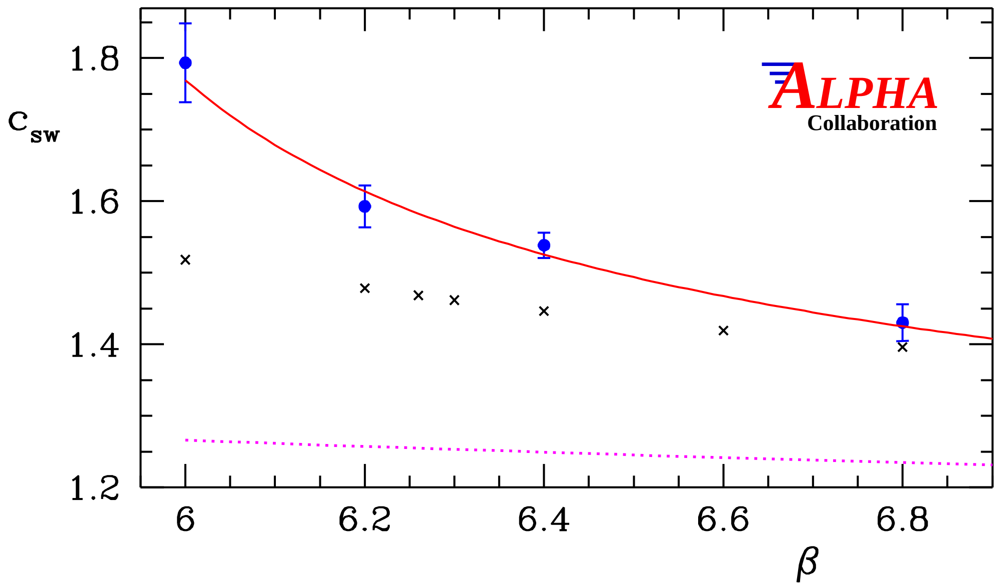
$$m_1 = m_2 \quad \text{and} \quad m_2 = m_3$$

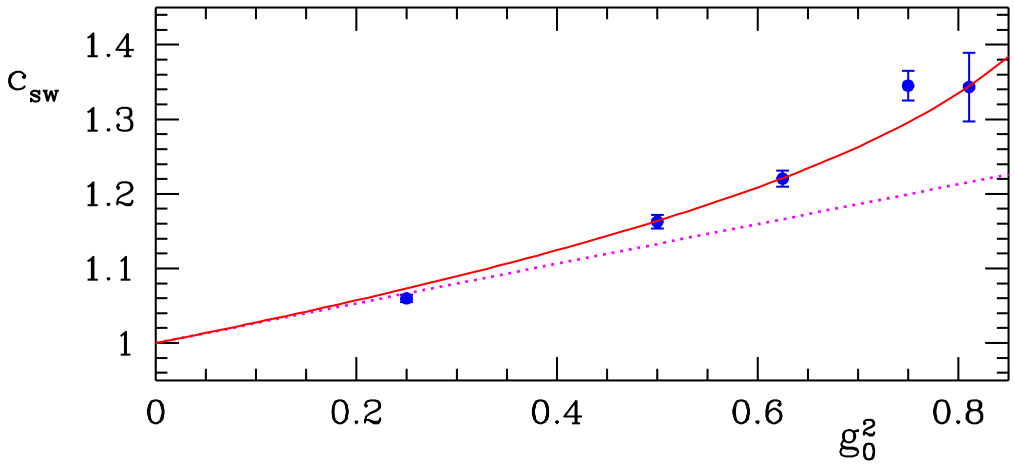
For any other configuration Symanzik's analysis then guarantees that $m = m_1 + \mathrm{O}(a^2)$

Different "improvement conditions" yield values of c_{sw} and c_{A} that differ by terms of $\mathrm{O}(a)$

$\Rightarrow$ improvement conditions should be chosen with care to avoid an artificial enhancement of $\mathrm{O}(a^2)$ cutoff effects

Results for c_{sw} in quenched QCD





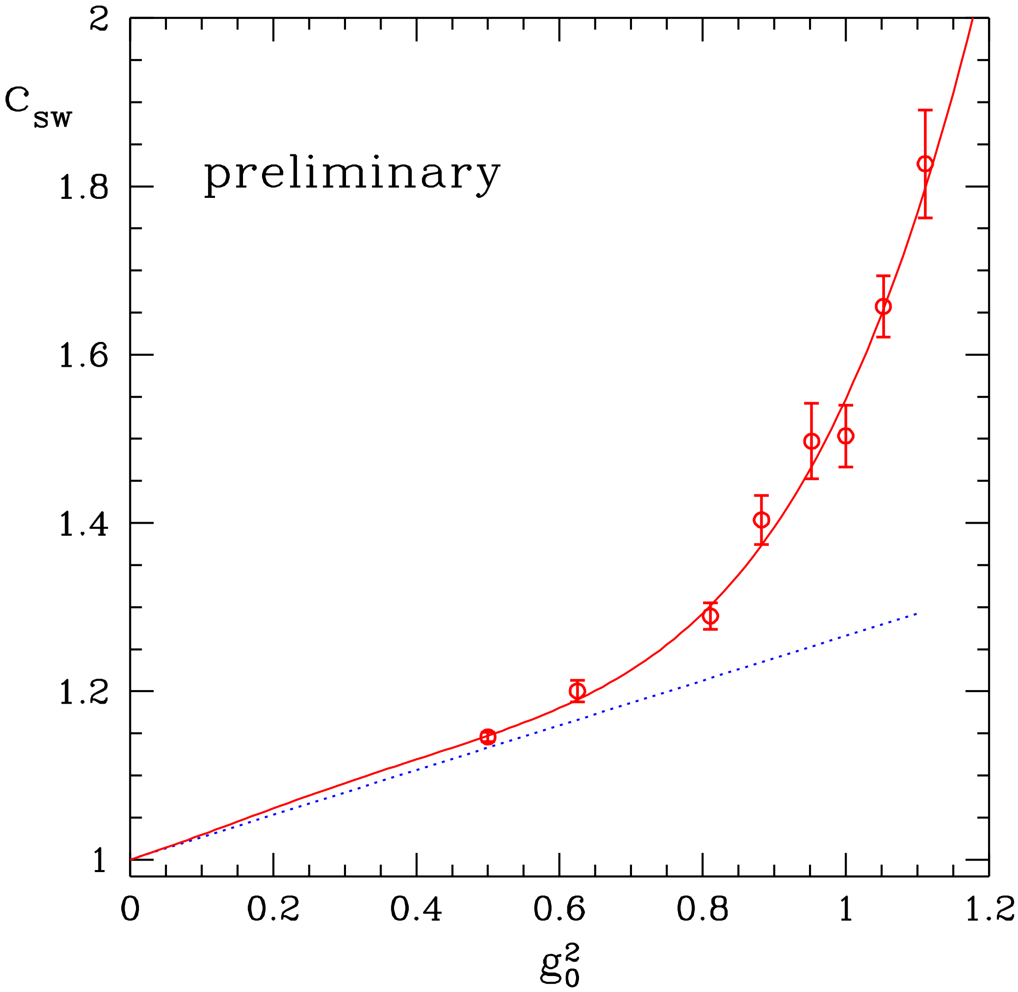
$$c_{\mathrm{sw}} = \frac{1 - 0.656\, g_0^2 - 0.152\, g_0^4 - 0.054\, g_0^6}{1 - 0.922\, g_0^2}, \qquad 0 \le g_0 \le 1$$

This expression is now taken as the definition of the improved action (Petronzio et al, Schierholz et al, UKQCD, . . .)

Extension to full QCD with $N_{\mathrm{f}} = 2$ dynamical quarks

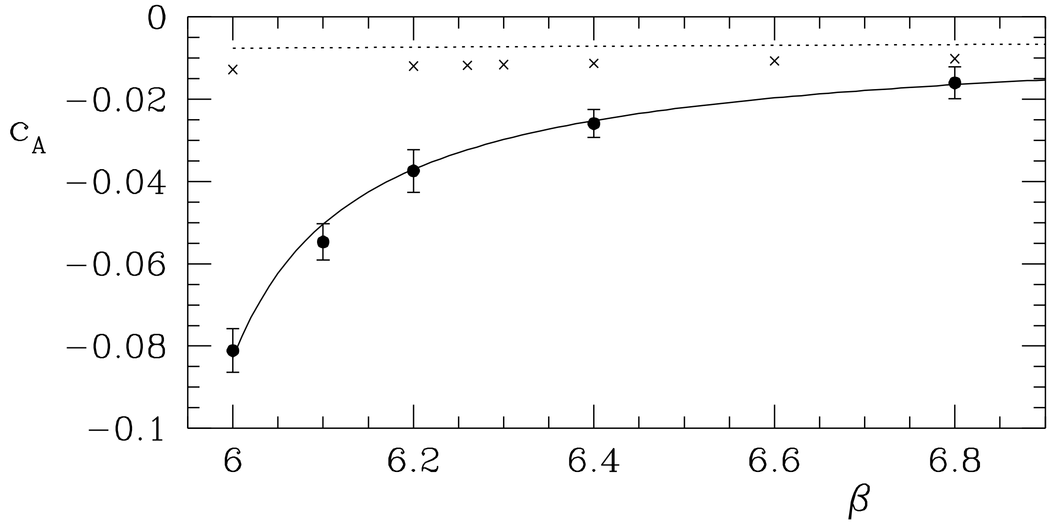
(K. Jansen & R. Sommer, work in progress)

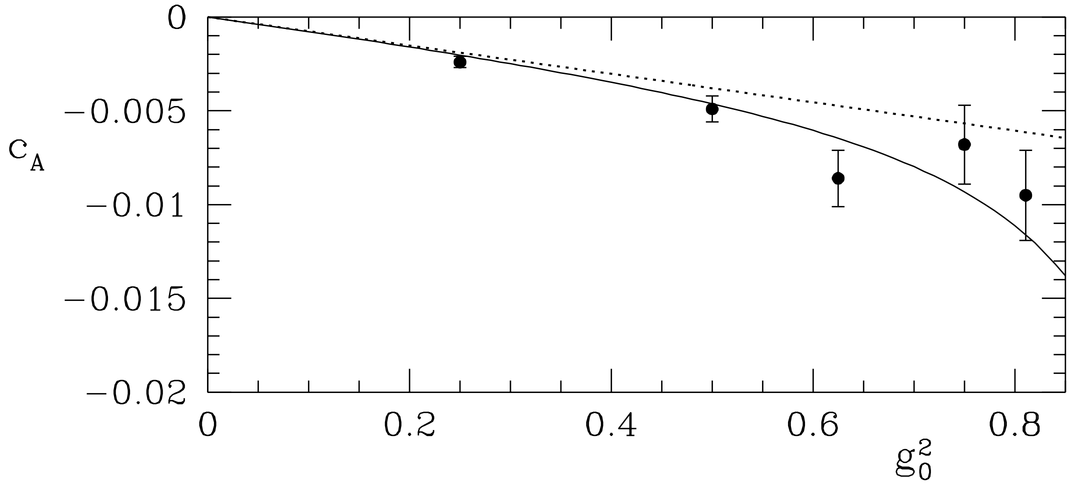
ALPHA Collaboration



(shaded region corresponds to $a \geq 0.1$ fm)

Results for c_{A} in quenched QCD

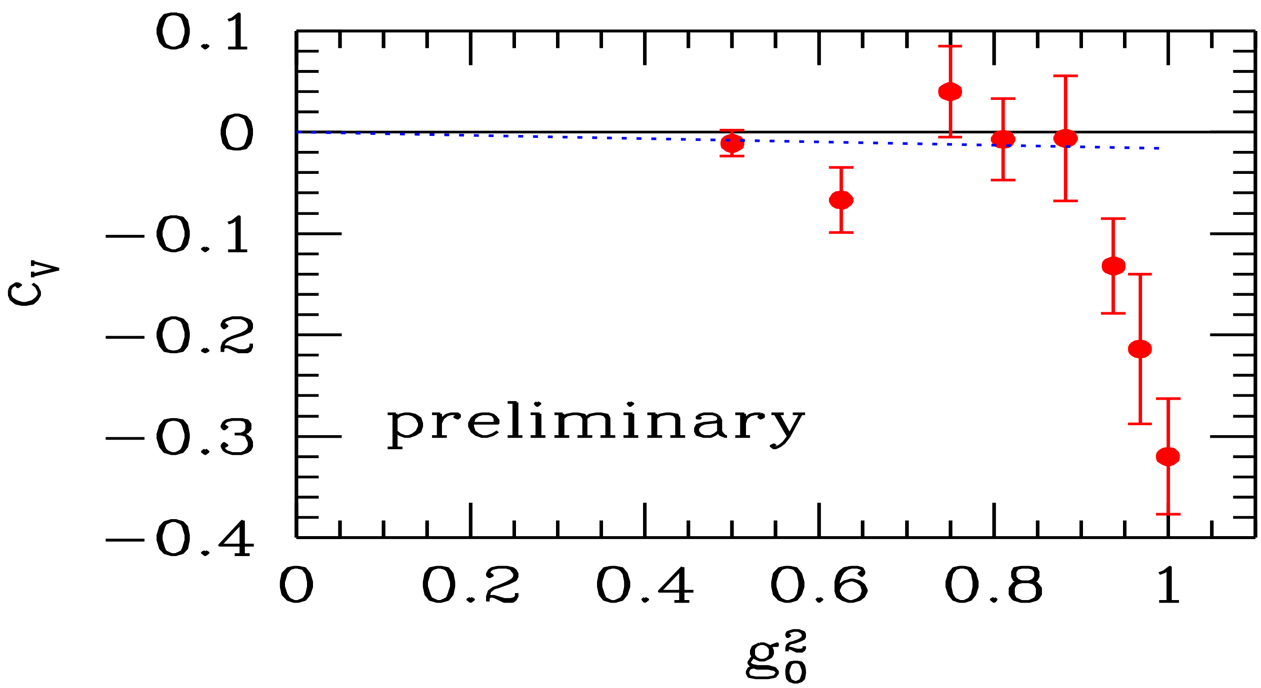




$$c_{\mathrm{A}} = -0.00756 g_0^2 \frac{1 - 0.748\, g_0^2}{1 - 0.977\, g_0^2}, \qquad 0 \leq g_0 \leq 1$$

Results for $c_{\rm V}$ in quenched QCD

(M. Guagnelli and R. Sommer, work in progress)



$$(V_{\rm I})^a_\mu = V^a_\mu + c_{\rm V}\partial_\nu T^a_{\mu\nu}, \quad T^a_{\mu\nu} = i\overline{\psi}\sigma_{\mu\nu}\tfrac{1}{2}\tau^a\psi$$

To 1-loop: $c_{\rm V} = -0.01633 g_0^2 + \dots$

SF Normalization conditions for composite operators

$\mathcal{O}^a = a^6 \sum_{\mathbf{u},\mathbf{v}} \overline{\psi}(u)\gamma_5 \frac{1}{2}\tau^a \psi(v), \qquad u_0 = v_0 = 0$

Correlation functions:

$$f_{\rm V}^{\rm R}(x_0) = \frac{a^3}{6L^6} \sum_{\mathbf{x}} i\epsilon^{abc} \langle \mathcal{O}'^a (V_{\rm R})_0^b(x) \mathcal{O}^c \rangle,$$

$$f_{\rm P}^{\rm R}(x_0) = -\frac{a^3}{3L^3} \sum_{\mathbf{x}} \langle (P_{\rm R})^a(x) \mathcal{O}^a \rangle,$$

$$f_{\rm AA}^{\rm R}(x_0) = \frac{a^6}{6L^6} \sum_{\mathbf{x},\mathbf{y}} \epsilon^{abe}\epsilon^{cde} \langle \mathcal{O}'^d (A_{\rm R})_0^a(x) (A_{\rm R})_0^b(y) \mathcal{O}^c \rangle,$$

$$f_1 = -\frac{1}{3L^6} \langle \mathcal{O}'^a \mathcal{O}^a \rangle.$$

Normalization conditions:

$$f_{\rm V}^{\rm R}(T/2) = f_1, \ T = 2L,$$

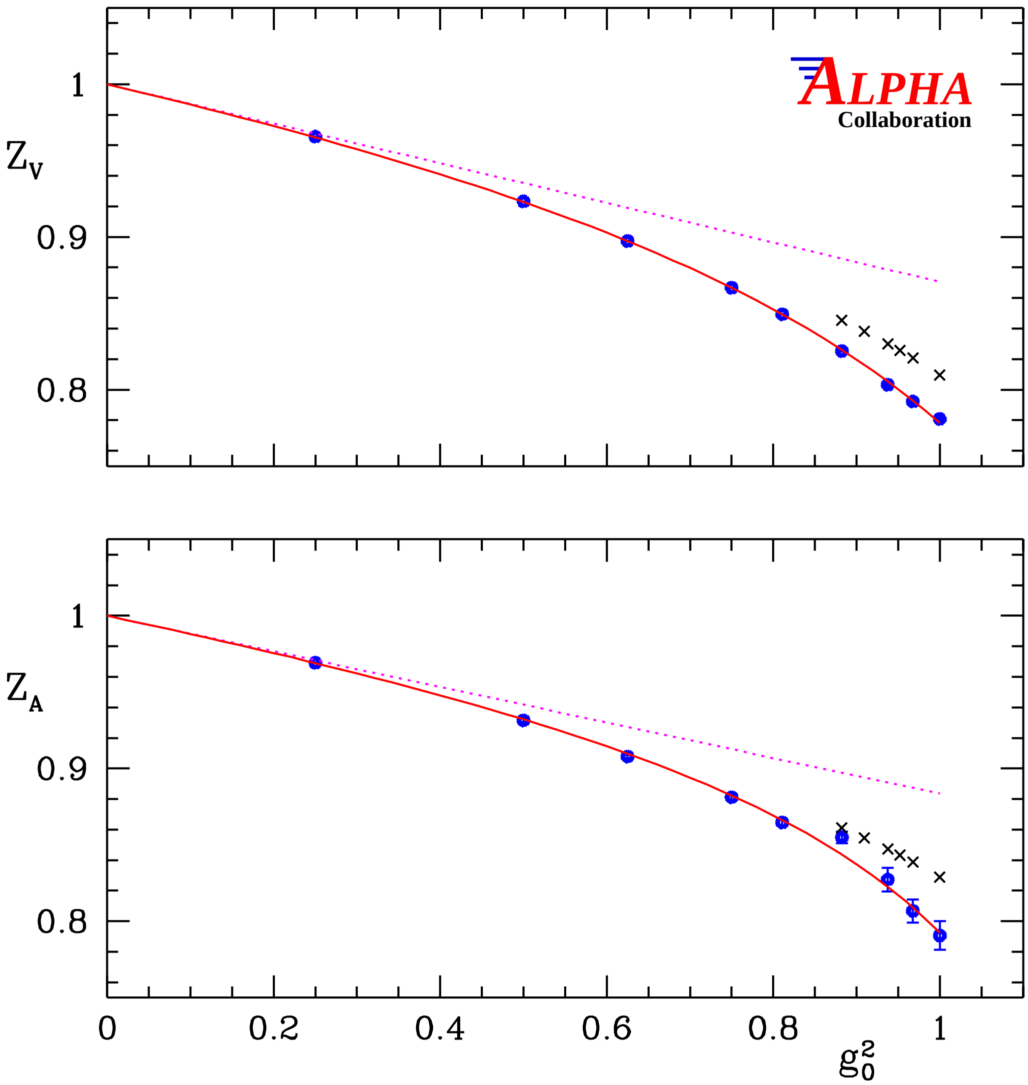
$$f_{\rm AA}^{\rm R}(2T/3, T/3) = f_1, \ \text{at } m_{\rm q} = 0, \ T = 9L/4,$$

$$f_{\rm P}^{\rm R}(T/2)^2 = f_1, \ \text{at } m_{\rm q} = 0.$$

First two conditions from consideration of Ward identities
Third condition completes definition of SF-running mass

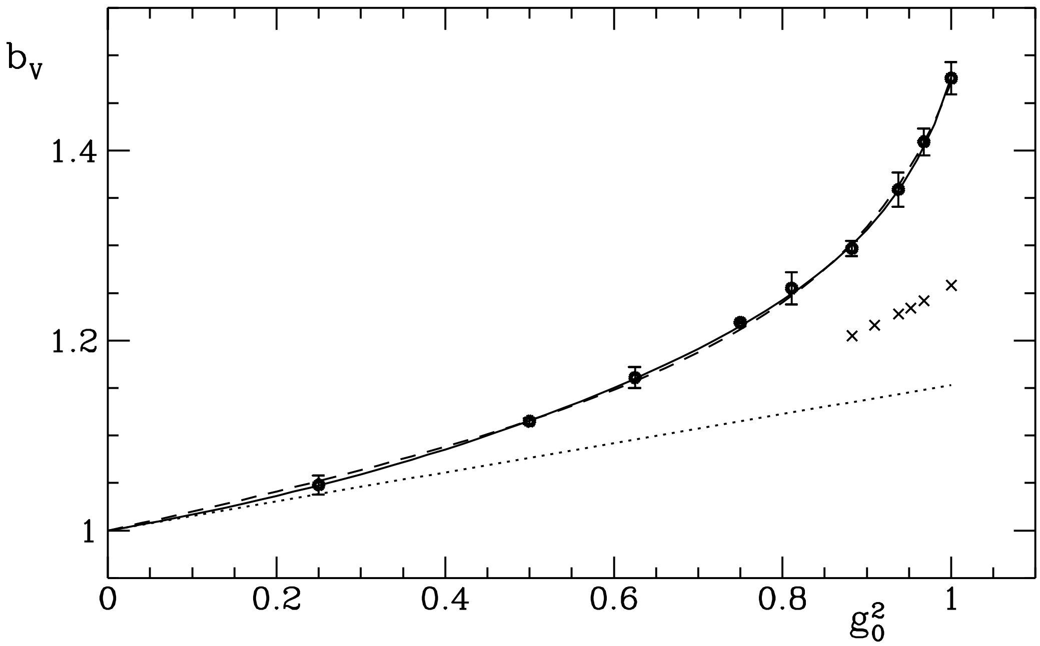
Non-perturbative determination of $Z_{\rm A}, Z_{\rm V}$ for $g_0^2 \leq 1$

(valence approximation)



Lüscher, Sint, Sommer, Wittig ('96)

Results for b_{V} in quenched QCD



$$b_{\mathrm{V}} = \frac{1 - 0.7613\, g_0^2 + 0.0012\, g_0^4 - 0.1136\, g_0^6}{1 - 0.9145\, g_0^2}, \qquad 0 \leq g_0 \leq 1$$

Tree level and one-loop improvement coefficients b

X	$b_X^{(0)}$	$b_X^{(1)}$
g	0	$0.012000(2) \times N_f$
m	$-\frac{1}{2}$	$-0.07217(2) \times C_F$
ζ	$-\frac{1}{2}$	$-0.06738(4) \times C_F$
A	1	$0.11414(4) \times C_F$
V	1	$0.11492(4) \times C_F$
P	1	$0.11484(2) \times C_F$
T	1	$-$

1-loop coeffs. (Sint & Sommer '96, Sint & Weisz '97)

CONCLUSIONS

- Schrödinger functional useful tool for implementing O(a)-improvement and non-perturbative renormalization

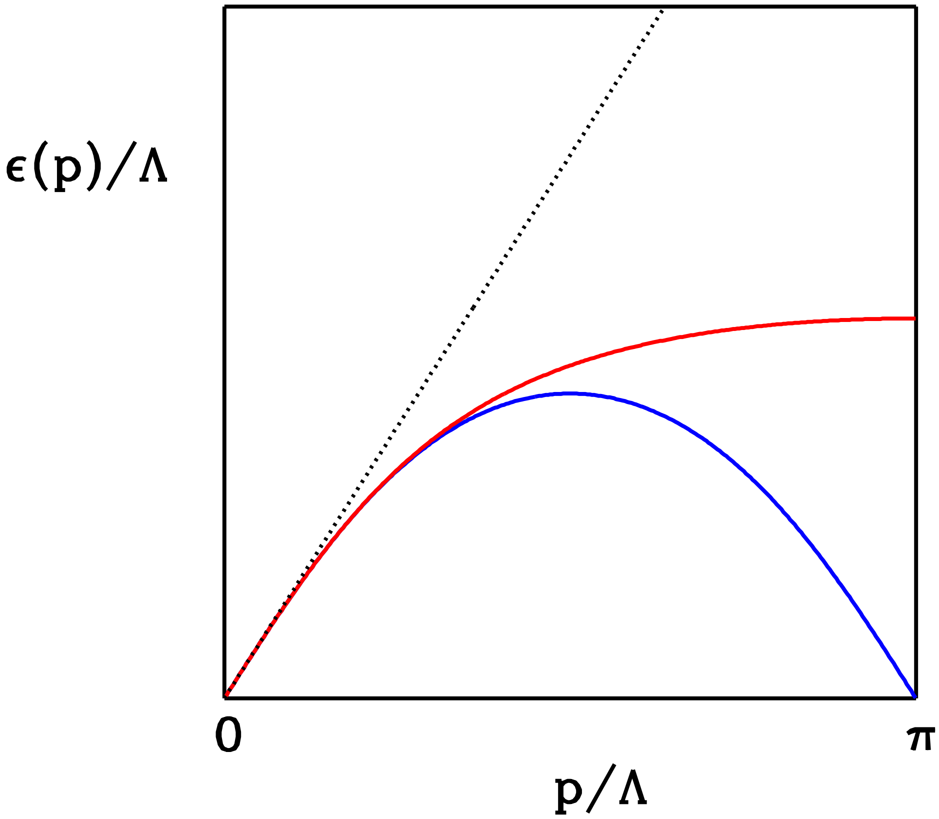
- $c_{\rm sw}$ and $c_{\rm A}$ determined non-perturbatively
  Done for $N_{\rm f} = 0$
  $N_{\rm f} = 2$ in progress (Jansen at LAT97)

- Chiral symmetry is accurately restored for $a \leq 0.1$ fm, provided the theory is fully O(a) improved

- Ongoing - Algorithm development
  - Computation of running quark mass
  - (Lüscher, Sint and Wittig at LAT97)

Why do we need the Wilson term?

Free quark energy-momentum relation; $\Lambda = 1/a$; $m_0 = 0$



HARDWARE

QUADRICS (APE) - developed by INFN $\rightarrow$ Alenia Spazio

present:

DESY/Zeuthen : two QF2 machines (and two Q1's)

QF2=256 Nodes $\times$ 50 MFlops/Node=12.8 GFlops

Maintained efficiency: $50 - 70\%$

future:

DESY early 1998: APE 1000 prototype ~ 64 GFlops

cf. CP-PACS (Japan)$\sim$300GFlops ($\sim 15 \cdot 10^6$ US$)

Columbia (USA) $\sim$800GFlops ($\sim 3 \cdot 10^6$ US$)

ALGORITHMS

QCD Lattice Action: $S = S[A] + \bar{\psi} Q \psi$

Dirac operator: $Q \sim D + m_0$

Integrate fermions (2 species):

$\langle \mathcal{O} \rangle = Z^{-1} \int d[A] \mathrm{det} Q^2 \exp(-S[A]) \mathcal{O}$

"Quenched approximation": $\mathrm{det} Q^2 \to 1$

IDEA (Lüscher 1994)

Approximating polynomial degree n: $P(Q^2) \sim Q^{-2}$

$\langle \mathcal{O} \rangle = Z^{-1} \int d[A] \mathrm{det} P^{-1} \cdot X \exp(-S[A]) \mathcal{O}$

$\sim$ simulation of n bosonic fields; $X = \det(P \cdot Q^2)$
(correction factor)

Competitive with Hybrid Monte-Carlo