Finite size techniques and the strong coupling constant

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LECTURE 2

- **The Alpha Collaboration**
- **Running coupling in Yang Mills theory**
- **The Schrödinger Functional**
- **The SF in QCD**

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ALPHA Collaboration

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Research Program

- Running coupling in QCD
- Quark masses
- Non-perturbative renormalization
- Symanzik improvement
- Simulation algorithms; dynamical quarks

Home Page: www.ifh.de/computing/parallel/ape/alpha/alpha.html

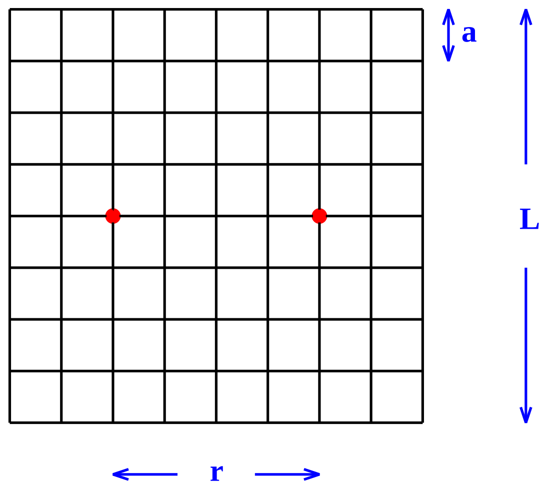
GOAL: Obtain $\alpha_{\overline{\mathrm{MS}}}(m_Z)$ to accuracy 1%

Require:

1. Accurate determination of a reference energy scale
2. A non-perturbatively defined $\alpha(q)$ **accurately** computable through numerical simulation over a wide range of q
3. Careful extrapolation to the continuum limit
4. Relation to $\alpha_{\overline{\mathrm{MS}}}$ to 2-loops

$$\alpha_{\overline{\mathrm{MS}}}(sq) = \alpha(q) + c_1(s)\alpha(q)^2 + c_2(s)\alpha(q)^3 + \ldots$$

Basic difficulty in this calculation is that low- and high-energy properties must be related to each other



Force between static quarks at distance r

$$F_{\mathrm{q\bar{q}}}(r) = \frac{4}{3}\frac{\alpha_{\mathrm{q\bar{q}}}}{r^2}$$

$\alpha_{\mathrm{q\bar{q}}}$ is a "running" coupling at momentum $q = 1/r$

$$\alpha_{\overline{\mathrm{MS}}} = \alpha_{\mathrm{q\bar{q}}} - 0.082 \times (\alpha_{\mathrm{q\bar{q}}})^2 + \ldots$$

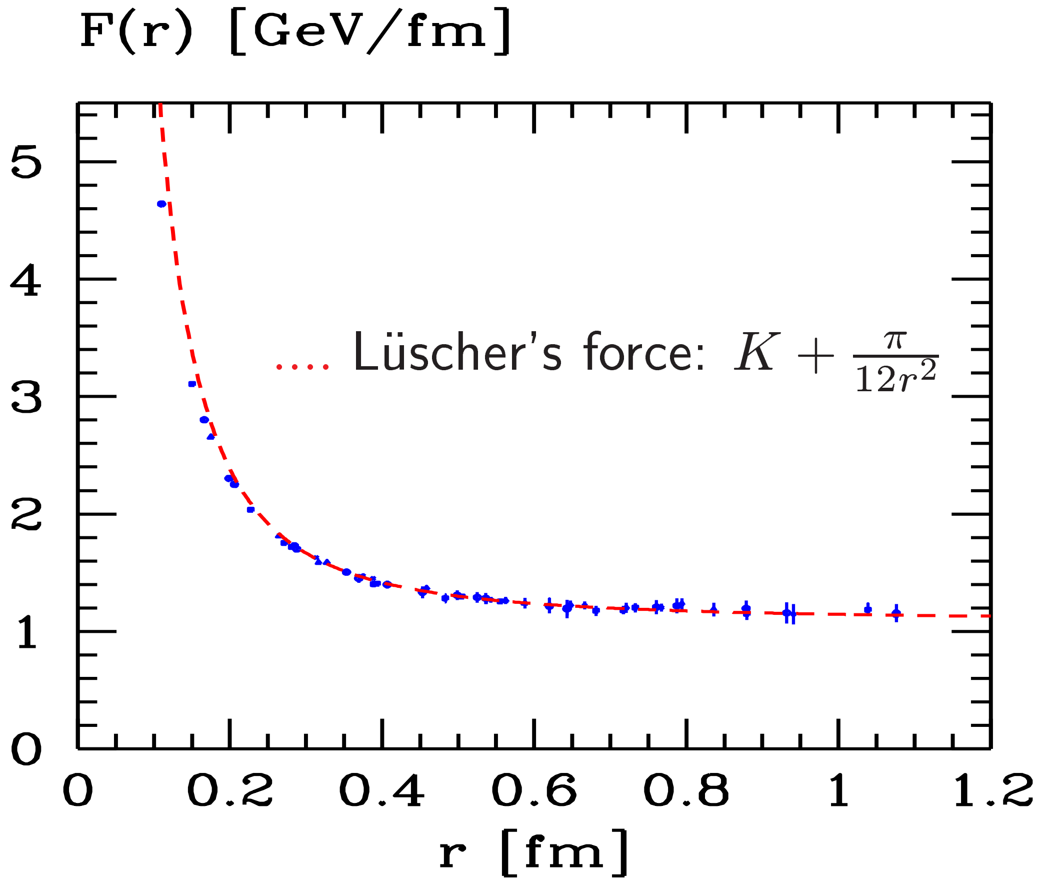
Example

32^4 lattice, $L = 2$ fm, $a = 0.06$ fm

$\Rightarrow r \geq 0.2$ fm, i.e. $q \leq 1$ GeV

Force between heavy quarks at distance $\mathbf{r}$

[Numerical data from R. Sommer, gauge group SU(2)]



Low energy scales

String tension: K

Sommer's scale r_0: distance where $r_0^2 F_{q\bar{q}}(r_0) = 1.65$

$-$ more easily computable than K

$r_0\sqrt{K} = 1.18(2)$ in SU(3) pure gauge theory

$r_0 \simeq 0.5$ fm (in Nature)

Solution: Define $\bar{g}^2(L)$ depending on physical extension L

Finite volume as probe of theory (instead of source of systematic error)

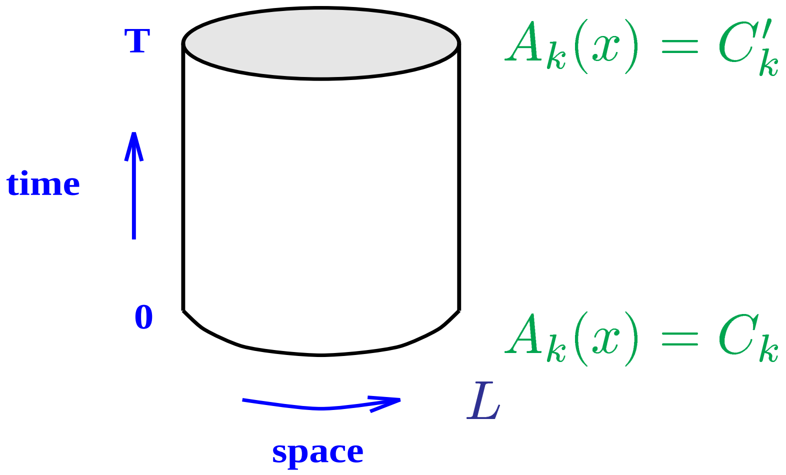
Important to make good choice of $\bar{g}$ such that:

- Non-perturbatively defined
- Accurately computable by numerical simulation
- Small cutoff effects
- Easy perturbative expansion

Difficult to fulfill!

Schrödinger functional

SU(N) Yang-Mills theory on $[0,T]\times S^1\times S^1\times S^1$



Schrödinger functional: $\mathcal{Z}[C',C] = \int D[A]\,\mathrm{e}^{-S[A]}$

quantum mechanical interp.: $= \langle C'|\mathrm{e}^{-\mathrm{H}T}\mathrm{P}|C\rangle$

H : Hamilton operator
P : projector on gauge invariant states

Lattice formulation

space-time $\rightarrow$ hyper-cubic lattice with spacing a

gauge field $\rightarrow$ field of matrices $U(x,\mu) \in \mathrm{SU}(N)$, associated with the lattice links

Periodic b.c. in all space directions, and at $x_0 = 0$

$$U(x,k)|_{x_0=0} = \mathcal{P}\exp\left\{a\int_0^1 \mathrm{d}t\, C_k(\mathbf{x}+(1-t)a\hat{\mathbf{k}})\right\}$$

Wilson plaquette action:

$$S[U] = \frac{1}{g_0^2}\sum_{\text{plaquettes } p} w(p)\,\mathrm{tr}\{1-U(p)\}$$

Lattice regularized Schrödinger functional:

$$\mathcal{Z}[C',C] = \int \prod_{x,\mu} \mathrm{d}U(x,\mu)\,\mathrm{e}^{-S[U]}$$

Well defined gauge invariant functional of C and C'

Perturbation theory = expansion about the "background field" $B_\mu(x)$ satisfying bc's with minimal action

Abelian boundary fields (chosen by Alpha Collab.)

$$C_k = \frac{i}{L} \times \mathrm{diag}\{\phi_1, \phi_2, \ldots, \phi_N\}, \qquad k = 1, 2, 3,$$

and similarly C', with angles ϕ_α satisfying

$$\phi_1 < \phi_2 < \ldots < \phi_N, \quad \phi_N - \phi_1 < 2\pi, \quad \sum_{\alpha=1}^{N} \phi_\alpha = 0$$

find a constant colour-electric background field

$$B_0 = 0, \qquad B_k = \frac{x_0}{T}(C'_k - C_k) + C_k$$

Stability theorem: For boundary values above, B is unique global minimum of the action

Renormalization of the SF

Space-time boundaries $\Rightarrow$ additional UV divergences – cancelled by adding boundary counterterms:

$$\int \mathrm{d}^3\mathbf{x}\,\{\mathcal{L}_\mathrm{b}(x)|_{x_0=0} + \mathcal{L}'_\mathrm{b}(x)|_{x_0=T}\}$$

to the action $S[A]$ (Symanzik, 1981)

$\mathcal{L}_\mathrm{b}(x)$ = linear combination of local fields of dimension $d \leq 3$, invariant under space & internal symmetries

In the pure gauge theory no such fields:

$\Rightarrow$ the effective action $\Gamma[B] - \Gamma[0] = \ln(\mathcal{Z}[B]/\mathcal{Z}[0])$ is finite after renormalization of the coupling exactly as in infinite volume.

Definition SF running coupling

Define a renormalized coupling $\bar{g}^2(L)$ through

$$\Gamma[B] = \frac{1}{g_0^2}\Gamma_0[B] + \Gamma_1[B] + \ldots \propto \frac{1}{\bar{g}^2}$$

for some particular C and C'

Find a good choice is a constant colour-electric field:

$$(\phi_1, \phi_2, \phi_3) = \tfrac{1}{3}(-\pi, 0, \pi) + \tfrac{1}{2}(2, -1, -1)\,\eta$$

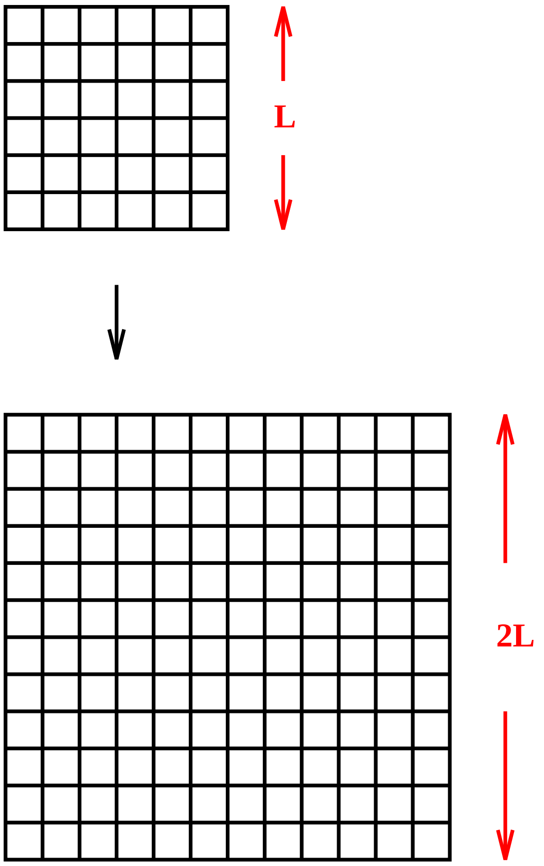
[and a similar expression for $(\phi_1', \phi_2', \phi_3')$]

Set $T = L$ and define

$$\bar{g}^{-2}(L) = \frac{1}{12\pi} \frac{\partial \Gamma}{\partial \eta}\bigg|_{\eta=0}$$

Fulfills all requirements mentioned previously

Computation of the scale evolution of $\bar{g}^2(L)$



choose lattice size L/a and
bare coupling g_0

compute $\bar{g}^2(L) \rightarrow \mathrm{result}$

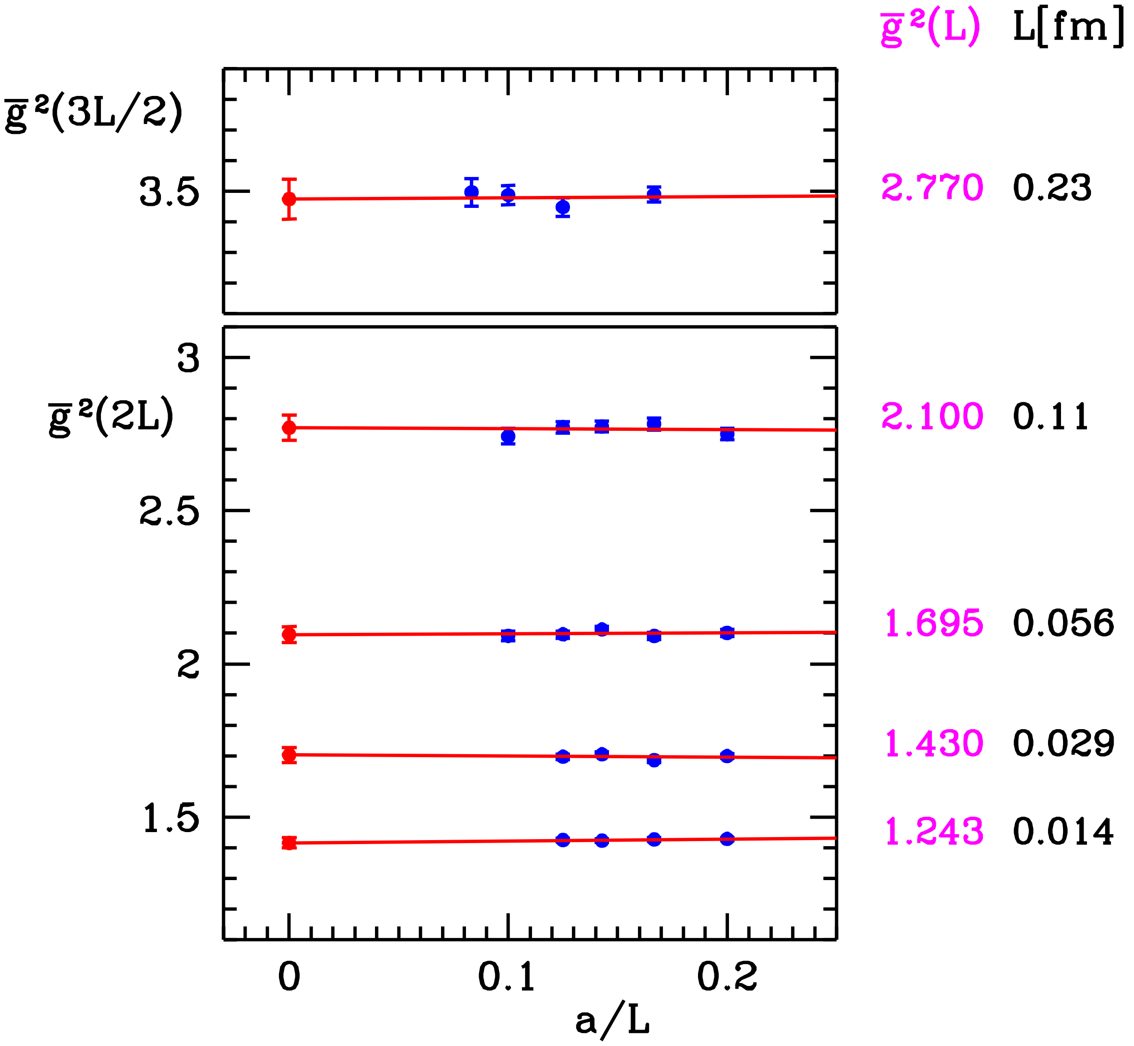
keep g_0 fixed

compute $\bar{g}^2(2L)$

Chose g_0 for bigger value of L/a giving same result
Repeat....finally extrapolate to continuum limit

Repeat procedure with another value of result
Highest and lowest energies differ by factor $2^{\# of steps}$

Taking the continuum limit à la Symanzik



Evolution of $\bar{g}^2(L)$ (SU(3)) in the continuum limit

s	$\bar{g}^2(L)$	$\bar{g}^2(sL)$
2	1.243	1.416(16)
2	1.430	1.703(24)
2	1.695	2.095(25)
2	2.100	2.771(41)
$3/2$	2.770	3.474(65)

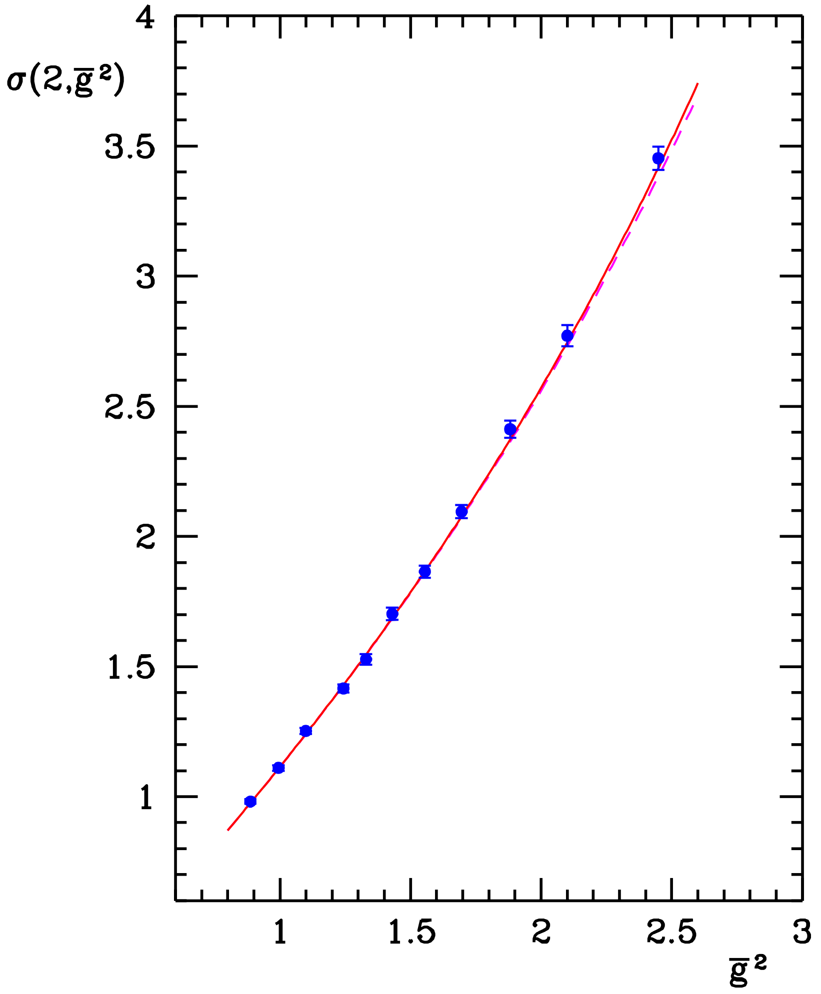
Evolution steps can be combined

$\Rightarrow$ obtain evolution of $\bar{g}^2(L)$ from 1.243 to 3.475, corresponding to a total scale factor of about 25

Above data 1993; new data achieves scale factor 64

see Lüscher LAT97

Th SF step scaling function in Pure SU(3) Yang-Mills Theory



$L_{\rm max} \equiv$ value of L where $\bar{g}^2(L) = 3.480$

$L/L_{\rm max}$	$\bar{g}^2(L)$
1.000	3.480
0.6642	2.770(39)
0.3320	2.100(30)
0.1647	1.695(25)
0.0836	1.430(24)
0.0399	1.243(21)

How large is $L_{\rm max}$ in physical units?

Computation of $L_{\rm max}/r_0$

Both factors in

$$L_{\rm max}/r_0 = (L_{\rm max}/a)(r_0/a)^{-1}$$

are functions of g_0 that can be determined separately

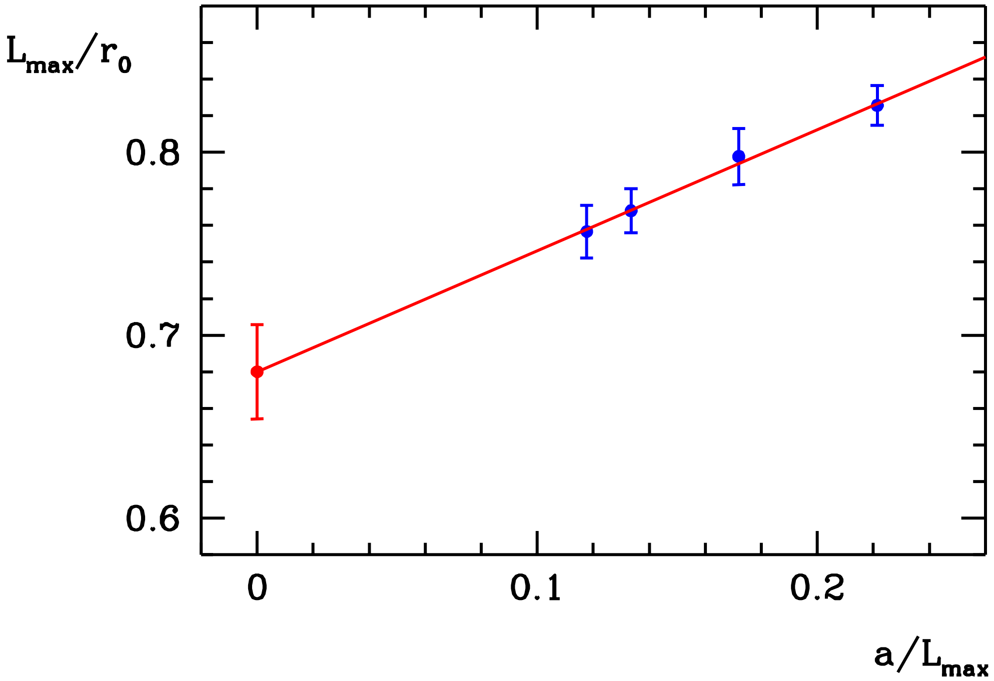
- Choose g_0 and compute r_0/a on a large lattice, avoiding finite-volume effects

- At the same g_0 compute $\bar{g}^2(L)$ for a range of L/a and find $L_{\rm max}/a$ through interpolation

Results are not independent of g_0, but

$$L_{\rm max}/r_0 = \text{constant} + \mathrm{O}(a/L_{\rm max})$$

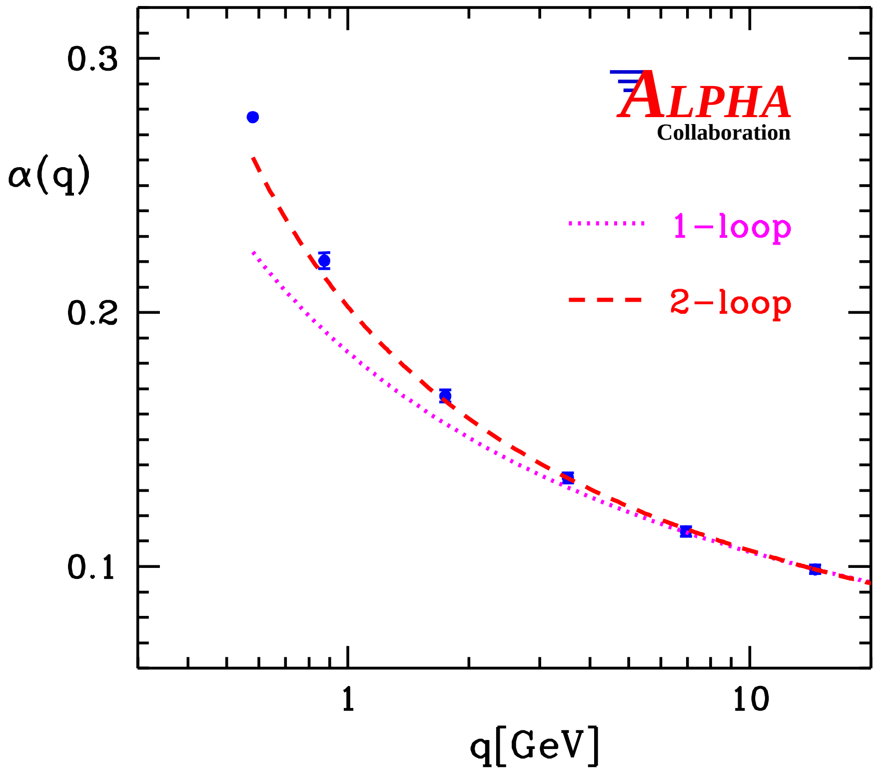
Extrapolation of L_{max}/r_0 to the continuum limit



$$L_{\mathrm{max}}/r_0 = 0.680(26) \quad \Rightarrow \quad L_{\mathrm{max}} = 0.34 \text{ fm}$$

The SF running coupling in Pure SU(3) Yang-Mills Theory ('93 data)

$$\alpha(q) = \frac{\bar{g}^2(L)}{4\pi}, \qquad q = 1/L$$



The SF running coupling in Pure SU(3) Yang-Mills Theory ('97 data)

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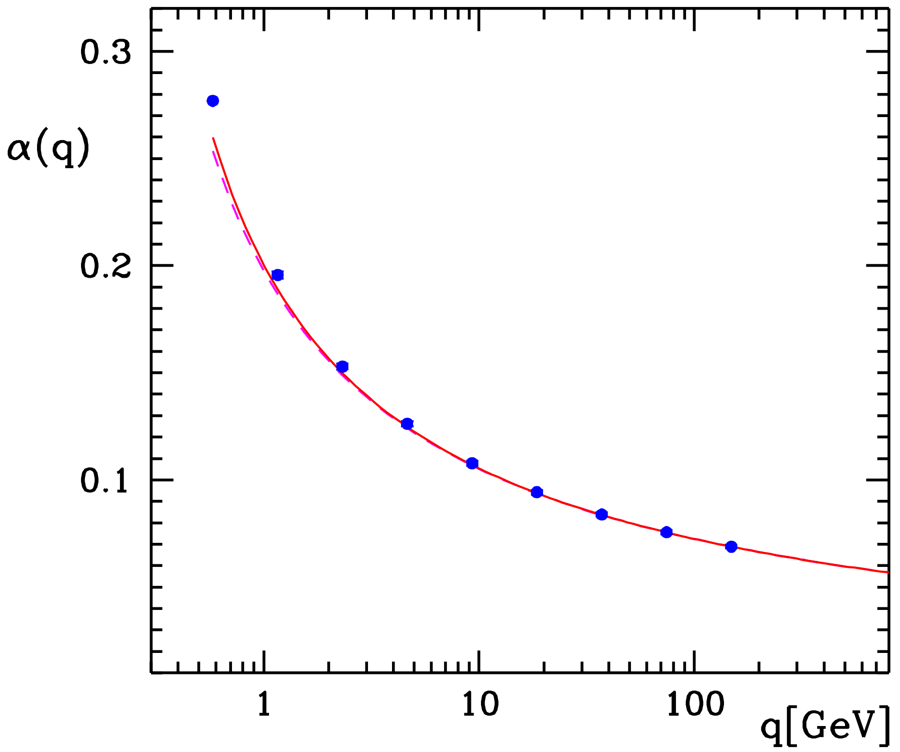

curves: 2-,3- loop evolution

Conversion to the $\overline{\mathrm{MS}}$ scheme

2-loop formula (Bode, Wolff, P.W. '97)

$$\alpha_{\overline{\mathrm{MS}}}(q) = \alpha(sq) + 0.27\alpha^3(sq) + \mathrm{O}(\alpha^4), \qquad s = 0.48811,$$

applies at large momenta q

e.g. if we choose

$$q = 75 \times r_0^{-1} \simeq 30\ \mathrm{GeV},$$

corr. to $L = 1/(sq) = 0.027 \times r_0 = 0.040 \times L_{\mathrm{max}}$,
obtain:

$$\alpha_{\overline{\mathrm{MS}}}(q) = 0.0986(14)(10) \qquad \text{(stat)}(\alpha^3),\ (1\text{–loop})$$

$$\alpha_{\overline{\mathrm{MS}}}(q) = 0.0989(14)(\ \ 1) \qquad \text{(stat)}(\alpha^4),\ (2\text{–loop})$$

Schrödinger functional in full QCD

bc on the quark field at $x_0 = 0$ (Sint '95)

$$\tfrac{1}{2}(1+\gamma_0)\psi(x) = \rho(\mathbf{x}), \qquad \overline{\psi}(x)\tfrac{1}{2}(1-\gamma_0) = \bar{\rho}(\mathbf{x})$$

SF now also depends on bdy. vals. ρ, $\bar{\rho}$ and ρ', $\bar{\rho}'$

- Renormalization requires a rescaling $\rho \to Z\rho$, etc.

- Running coupling $\bar{g}^2$ defined as before, with $\rho = \ldots = \bar{\rho}' = 0$

- Numerical simulations at zero quark mass are practical for $L \leq 0.5$ fm, because the finite volume provides an infrared cutoff

- $\Rightarrow$ $\bar{g}^2$ is computable in QCD with massless quarks

- The quark mass and operator renormalizations can also be studied in this framework

CONCLUSIONS

- Finite size scaling analysis works well in pure Yang Mills Theory
- Sources of systematic errors under control, in particular approach to continuum limit consistent with Symanzik's
- Schrödinger functional useful quantity for numerical and analytical studies in gauge theories
- Definition of SF extended to include fermions