Finite size techniques and the strong coupling constant

Peter Weisz (Max-Planck-Institute, Münich)

LECTURE 1

- **Running couplings**
- **Conventional wisdom**
- **The 2-d O(n) sigma model**
- **Recursive finite size scaling technique**
- **Approach to continuum limit**

LECTURE 2

- The Alpha Collaboration
- Running coupling in Yang Mills theory
- The Schrödinger Functional
- The SF in QCD

LECTURE 3

- Chiral symmetry
- Symanzik O(a) improved action
- Improved operators
- Non-perturbative renormalization

RUNNING COUPLING

many definitions e.g.

$$\frac{\sigma(e^+e^-\to\text{hadrons})}{\sigma(e^+e^-\to\mu^+\mu^-)} \equiv 3\sum_f Q_f^2(1+\frac{1}{\pi}\alpha_R(q))$$

at HE: $\alpha_R(q) = \alpha_{\overline{\rm MS}}(q) + c_1\alpha_{\overline{\rm MS}}(q)^2 + c_2\alpha_{\overline{\rm MS}}(q)^3 + \dots$

PT $\to$ **evolution** at HE but **not its value** at a given q

$\alpha_R(q)$ in principle computable in LATTICE QCD

Tune parameters at low energy by fixing scales: $f_\pi, m_\rho, ..$

$\to$ amplitudes determined at all energies e.g.

$\int_x e^{iqx}\langle j_\mu^{\rm em}(x) j_\nu^{\rm em}(0)\rangle$ with q in units of f_π $\to \alpha_R(q)$

TWO COMPLEMENTARY PROGRAMS

P1: e.g ALPHA Collaboration

Q: HOW DOES A NON-PERT DEFINED α RUN?

According to PT at HE?

If so, when does 2-loop running set in?

GOAL: $\alpha_{\overline{\mathrm{MS}}}(m_Z)$ to **quotable error** 1%

P2: e.g. Fermilab

ESTIMATE $\alpha_{\overline{\mathrm{MS}}}(m_Z)$ FROM PRESENT SIMULATIONS

CONTINUUM LIMIT: Delicate - lattice spacing $a \to 0$

with m_ρ, f_π... fixed requires $g_0 \to g_{\mathrm{crit}}$

"CONVENTIONAL WISDOM"	"HERETICAL SCENARIO"
Non-trivial contin. limit	yes
Mass gap	yes
quark confinement	yes
$g_{\mathrm{crit}} = 0$	$g_{\mathrm{crit}} \neq 0$
asymptotic freedom	$g(\infty) \neq 0$

NO RIGOROUS PROOFS YET
EXPERIMENT WON'T DISTINGUISH

CONVENTIONAL WISDOM =
(plausible) WORKING HYPOTHESIS

NON-LINEAR O(N) σ-MODEL IN 2 D

Fields: $\sigma^a(x),\ a = 1, ..., N,\ \sigma^2 = 1$

Action: $S = \frac{1}{2g_0^2}\int d^2x(\partial_\mu\sigma^a)^2$

CONVENTIONAL WISDOM:

- asymptotically free: $g(q^2) \sim 1/b_0 \ln(q^2/\Lambda^2)$
- spectrum: 1 massive multiplet, mass m
- conservation laws $\rightarrow$ no particle production
- exact S-Matrix conjecture (Zamolodchikov^2)

$$S_{ik;jl}(p_1, p_2; p_1', p_2') = \delta(p_1 - p_1')\delta(p_2 - p_2')\sigma_2(\theta)[\delta_{ij}\delta_{kl} - \frac{2\pi i}{N-2}\{\frac{1}{i\pi-\theta}\delta_{ik}\delta_{jl} + \frac{1}{\theta}\delta_{il}\delta_{jk}\}] + (i \leftrightarrow k, p_1 \leftrightarrow p_2)$$

rapidities $p_j = m \sinh\theta_j,\ \theta = \theta_1 - \theta_2$

no bound states, "minimality": $\rightarrow \sigma_2(\theta)$

- exact m/Λ (P. Hasenfratz and F. Niedermayer)

Matrix elements of local operators $\mathcal{O}$:

$$\langle\Omega|\mathcal{O}|n; i_n, \theta_n, ...\rangle =$$
$$(m\sum_j e^{\theta_j})^{\Delta+s}(m\sum_k e^{-\theta_k})^{\Delta-s} f^{(n)}_{i_n,\ldots,i_1}(\mathcal{O}; \theta_n, \ldots)$$

Smirnov Axioms:

I.a (Watson's theorem):
$$f^{(n)}_{i_n,\ldots,i_1}(\mathcal{O}; \theta_n, \theta_{n-1}, \ldots) = S^{j_{n-1}j_n}_{i_n,i_{n-1}}(\theta_{n,n-1}) f^{(n)}_{j_n,\ldots,i_1}(\mathcal{O}; \theta_{n-1}, \theta_n, \ldots)$$

I.b (cyclicity):
$$f^{(n)}_{i_n,\ldots,i_1}(\mathcal{O}; \theta_n + 2\pi i, \theta_{n-1}, \ldots) = f^{(n)}_{i_{n-1},\ldots,i_1,i_n}(\mathcal{O}; \theta_{n-1}, \ldots, \theta_1, \theta_n)$$

II. $\mathrm{Res}_{\theta_n=\theta_{n-1}+i\pi}[f^{(n)}_{i_n,..,i_1}(\mathcal{O}; \theta_n, ..)] = \mathcal{T}[S]^{j_{n-2},..,j_1}_{i_n,\ldots,i_1} f^{(n-2)}_{j_{n-2},..,j_1}(\mathcal{O}; \theta_{n-2}, ..)$

Exact form factors of local operators for $N = 3$ e.g. :

$l = 1$: spin-field: σ^a,Noether current: $j^a_\mu = g_0^{-2}\epsilon_{abc}\sigma^b\partial_\mu\sigma^c$

$l = 0$: EM tensor: $\theta_{\mu\nu} = g_0^{-2}[\partial_\mu\sigma^a\partial_\nu\sigma^a - \frac{1}{2}\eta_{\mu\nu}\partial_\rho\sigma^a\partial_\rho\sigma^a]$,

topological density: $k = \frac{1}{8\pi}\epsilon_{abc}\epsilon_{\mu\nu}\sigma^a\partial_\mu\sigma^b\partial_\nu\sigma^c$

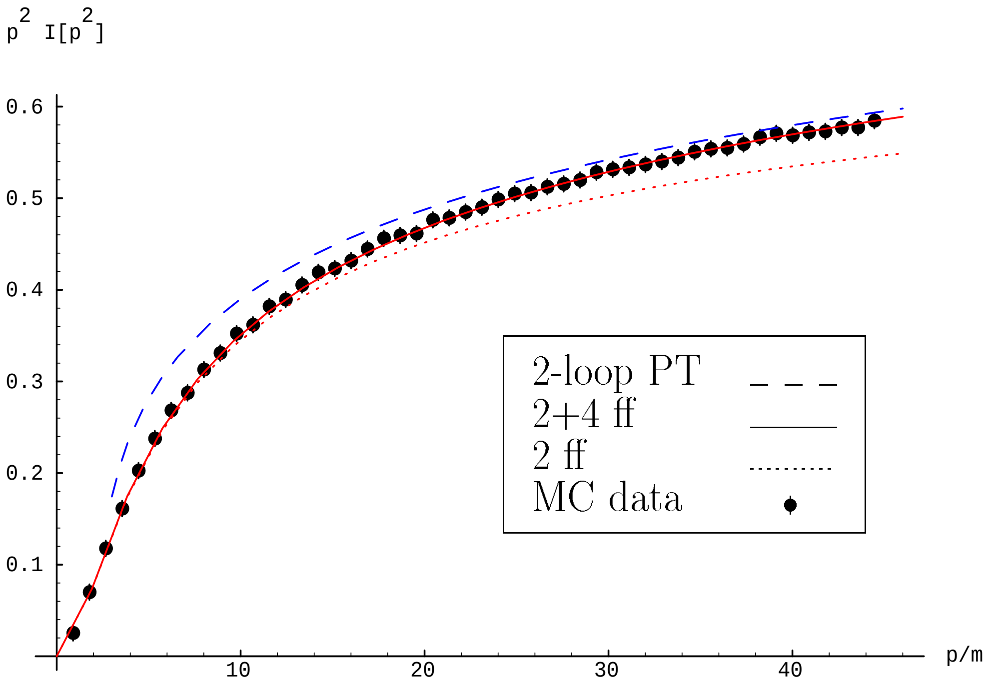
Results for 2-pt correlation functions e.g. :

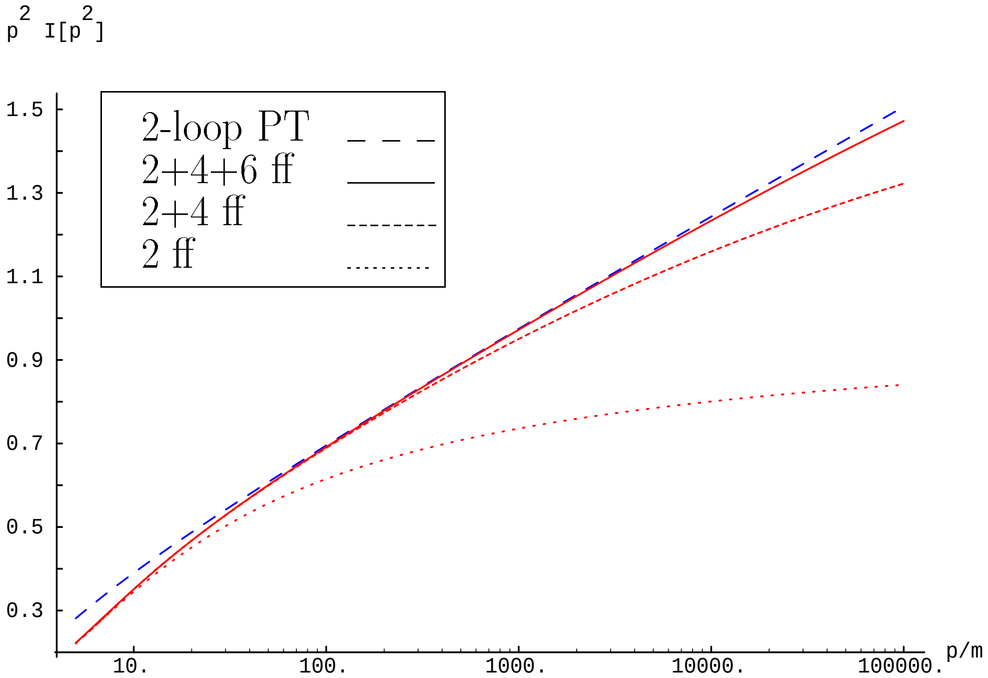
$$\int d^2x e^{ipx}\langle j^a_\mu(x) j^b_\nu(0)\rangle = \delta^{ab} I(p^2)(p_\mu p_\nu - \delta_{\mu\nu}p^2)$$

by doing partial sum over intermediate states: $1 = \sum_n |n\rangle\langle n|$

O(3) 2d σ-model current-current correlator

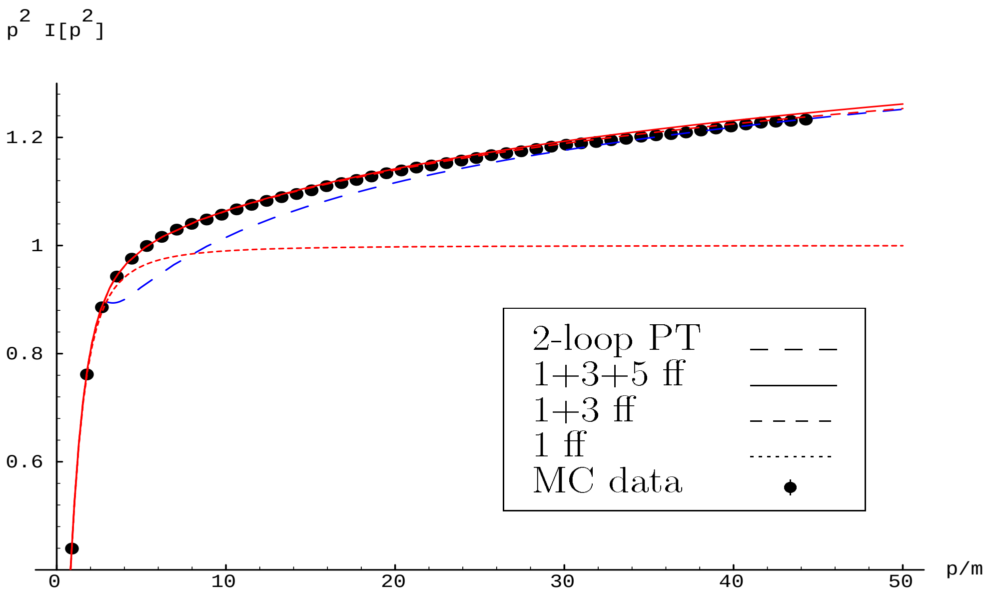
J. Balog and M. Niedermaier ('96)

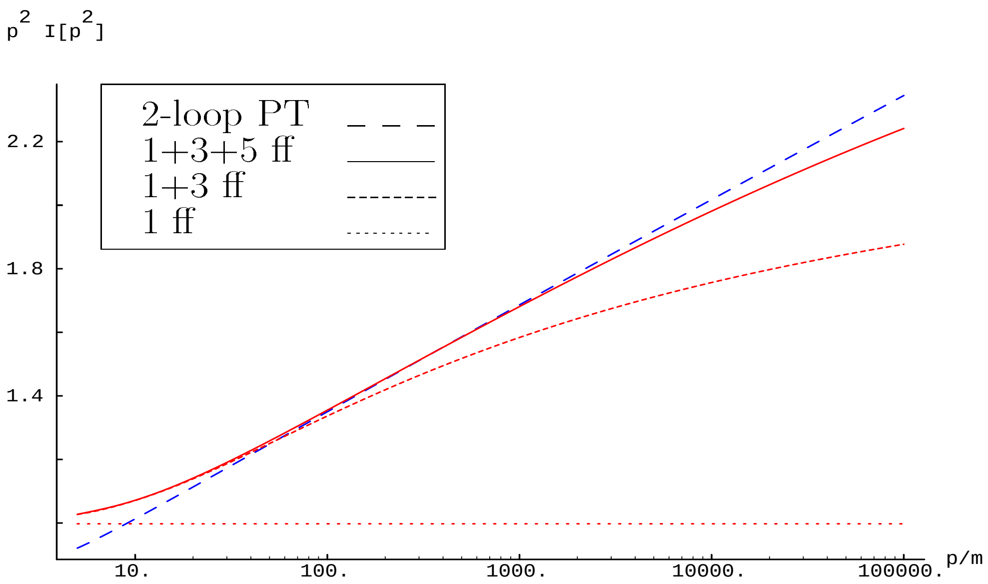




O(3) 2d σ-model spin-spin correlator

J. Balog and M. Niedermaier ('96)





Spectral decomposition: $I_{\mathcal{O}}(p^2) = \int_0^\infty d\mu \frac{\rho_{\mathcal{O}}(\mu)}{\mu^2+p^2}$

Spectral functions: $\rho_{\mathcal{O}}(\mu) = \sum_n \rho_{\mathcal{O}}^{(n)}(\mu)$

for $\mu \to \infty$: $\rho_{\mathcal{O}}^{(n)}(\mu) \sim \frac{A_{\mathcal{O}}^{(n)}}{\mu(\ln\mu)^{4-2l}}$

cf RPT: $\rho_1(\mu) \sim \frac{A_{\mathcal{O}}}{\mu}[1 + O(\frac{1}{\ln\mu}) + \ldots]$

$$\rho_0(\mu) \sim \frac{A_{\mathcal{O}}}{\mu}[\frac{1}{(\ln\mu)^2} + O(\frac{\ln\ln\mu}{(\ln\mu)^3}) + \ldots]$$

$$\rho_l^{(n)}(\mu)/\mu = \qquad = M_l^{(n)} Y_l^{(n)}(\frac{\ln(\mu/m)}{\xi_l^{(n)}})$$

Cluster properties relate odd and even n

Scaling hypothesis for $n \to \infty$: $Y_l^{(n)} \to Y_l$,

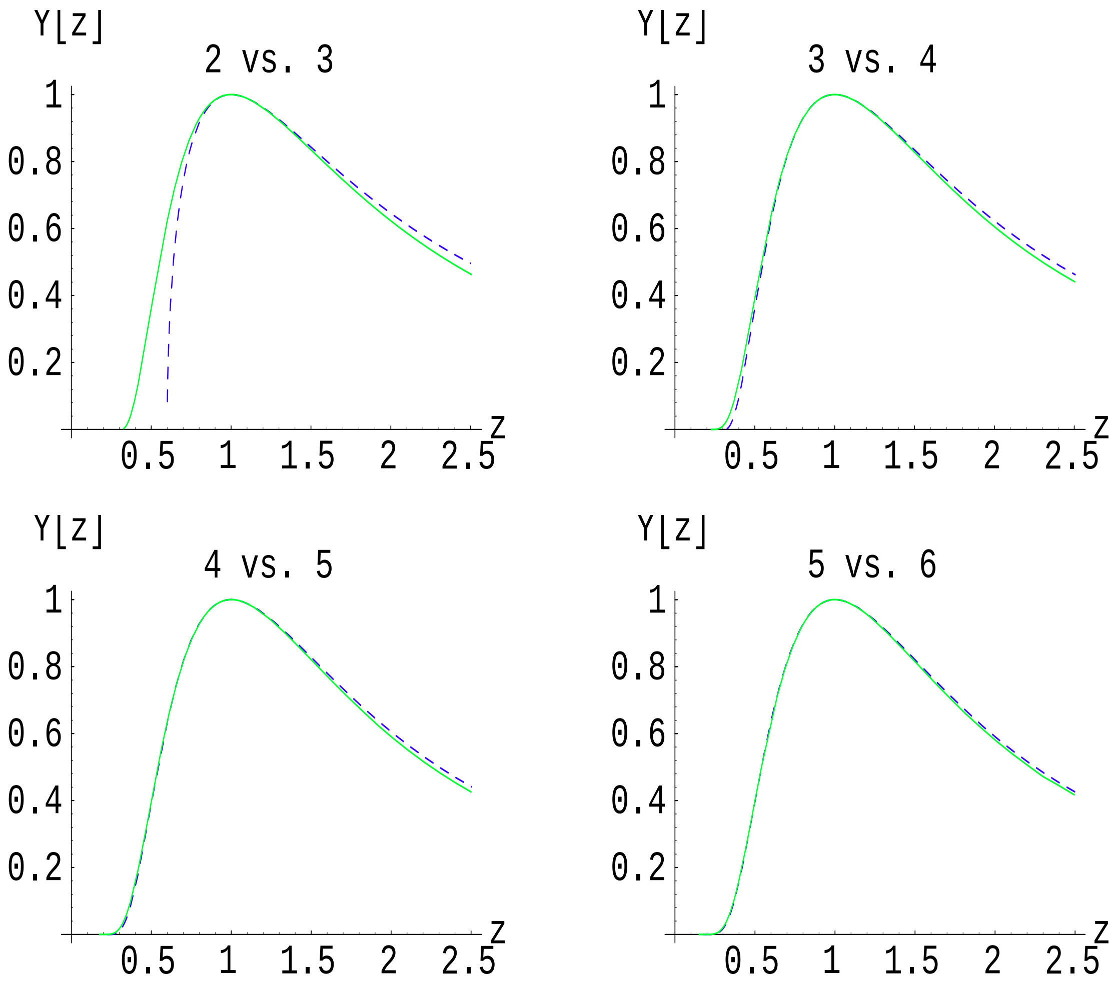
$$\xi_l^{(n)} \sim \xi_l n^{1+\alpha_l}, \quad M_l^{(n)} \sim M_l n^{\gamma_l}$$

consistency: $\alpha_0 = \alpha_1 = \alpha, \;\; \gamma_0 = 3 + 2\alpha, \;\; \gamma_1 = 1$

fit: $\alpha = 0.273$

O(3) 2d σ-model n particle spectral functions (Isospin 1)

J. Balog and M. Niedermaier ('96)



Consequences of scaling hypothesis

- Koba-Nielsen-Olesen scaling ('72) of multiparicle inclusive X-sections $P_n = \sigma_n(s)/\sigma_{\text{tot}}(s)$
  $\bar{n}P_n \sim \psi(n/\bar{n}) \quad s \to \infty$

  e.g. coupling to vector current
  $P_{2n} \propto \mu\rho_1^{(2n)}(\mu),\ \bar{n} \sim (\ln\mu)^{1/(1+\alpha)}$

- UV behaviour qualitatively consistent with PT

- Quantities accessible to PT reproduced to $\sim 1\%$

  e.g. central charge:
  $c = 1.603 + 0.194 + 0.072 + \ldots = 1.997$

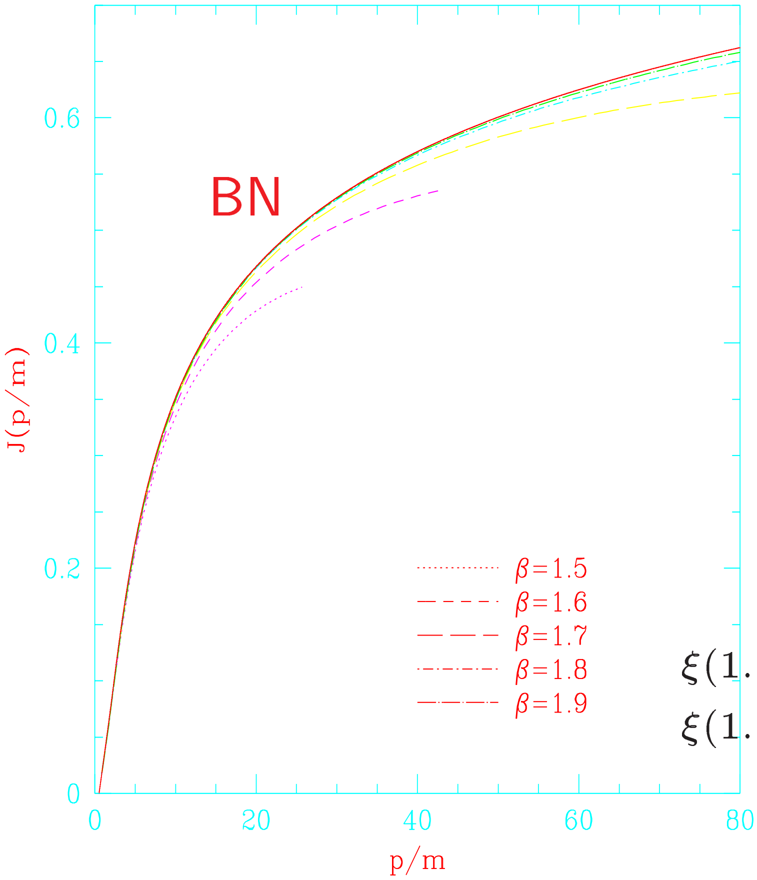
  and $A^{\text{EM}}/A^{\text{EM}}_{\text{PT}} = 0.99$

- Some normalizations not accessible to PT are fixed

  e.g. $A^{\text{curr}} = \frac{\pi}{4}A^{\text{spin}},\ A^{\text{EM}} = 4A^{\text{top}}$

Comparison of current-current correlators of the O(3) 2d σ-model and dodecahedron model

A. Patrascioiu and E. Seiler (June '97)



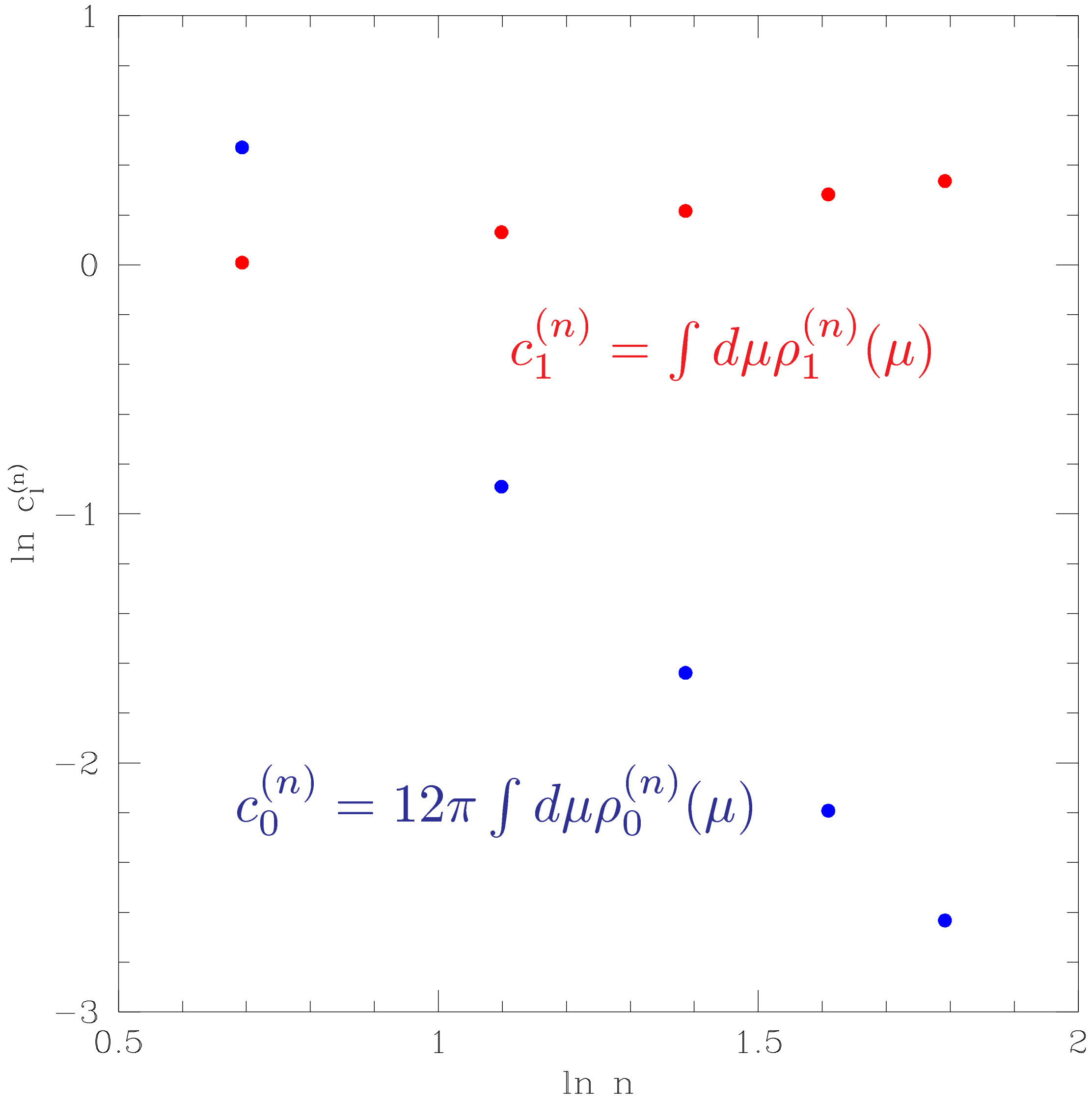
$\xi(1.8) = 64.67(8)$

$\xi(1.9) = 122.15(39)$

After normalizing the current operator in the dodecahedron model (20 points on S^2), the functions $J(p/m)$ fall on top of each other at correlation lengths ~ 65!

Patrascioiu and Seiler conclude: Continuum limit of O(3) and dodecahedron lattice models coincide
$\rightarrow$ O(3) not asymptotically free

Spectral function integrals in 2-d O(3) σ-model from Balog and Niedermaier



Patrascioiu and Seiler scenario requires $c_1^{(n)} \to 0$

In 4-dim. theories, such as QCD, running couplings

defined in infinite volume through e.g.

$$\int d^2 x e^{ipx} \langle j_\mu(x) j_\nu(0) \rangle$$

are extremely difficult to measure numerically over a wide range of energies while controlling systematic errors.

Without extrapolations need

$a \ll r \ll L = na$; typically presently $n \leq 48$

SOLUTION: Recursive finite size technique

Recursive finite-size technique

1. Define running coupling $\bar{g}^2(L)$ in finite volume

$*$ 2. Compute evolution of $\bar{g}^2(L)$ from small to large L recursively

3. Make contact with the physical low-energy scales at large L

4. At small L convert to the $\overline{\mathrm{MS}}$ scheme using perturbation theory

THE STEP SCALING FUNCTION

$\bar{g}^2(2L)$

$\leftarrow$ Continuum curve

$\bar{g}^2(L)$

$\uparrow$

some "admissible coupling"

Good choice of $\bar{g}(L)$ such that

- Non-perturbatively defined
- Accurately computable by numerical simulation
- Small cutoff effects
- Easy perturbative expansion

e.g. in the O(3) non-linear sigma model on an infinite cylinder:

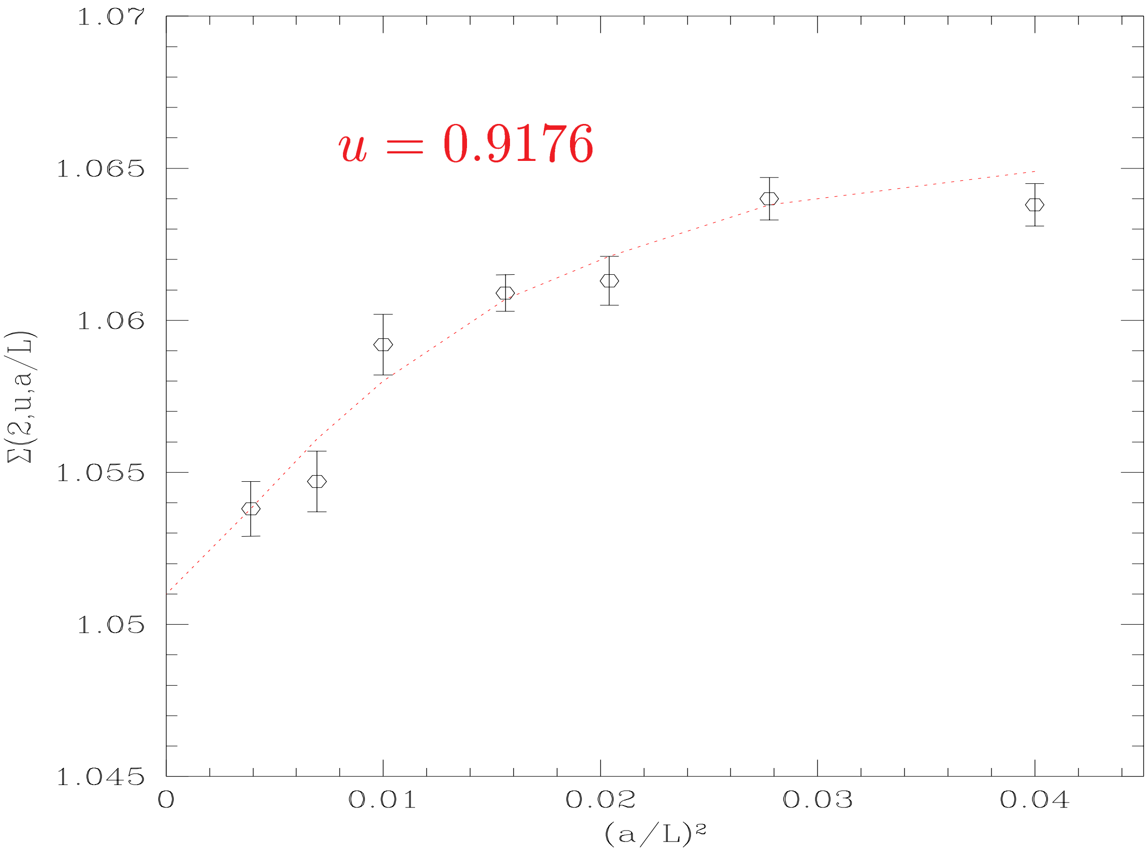
$\bar{g}(L)^2 = z(L) = M(L)L,$

$M(L) =$finite volume mass gap

(Lüscher, Weisz & Wolff, '91)

Approach to continuum limit of step scaling function in 2-d O(3) σ-model

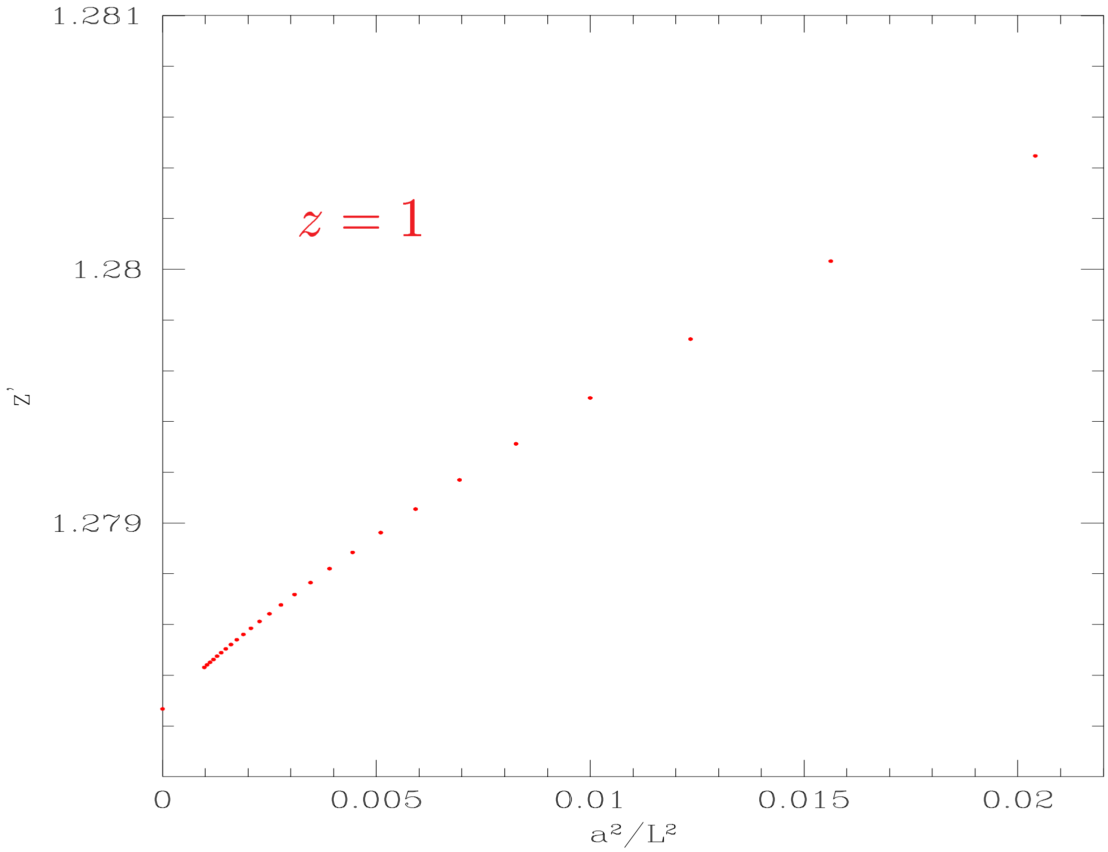
Standard action: $S = -g_0^{-2} \sum_{<i,j>} \sigma_i \cdot \sigma_j$



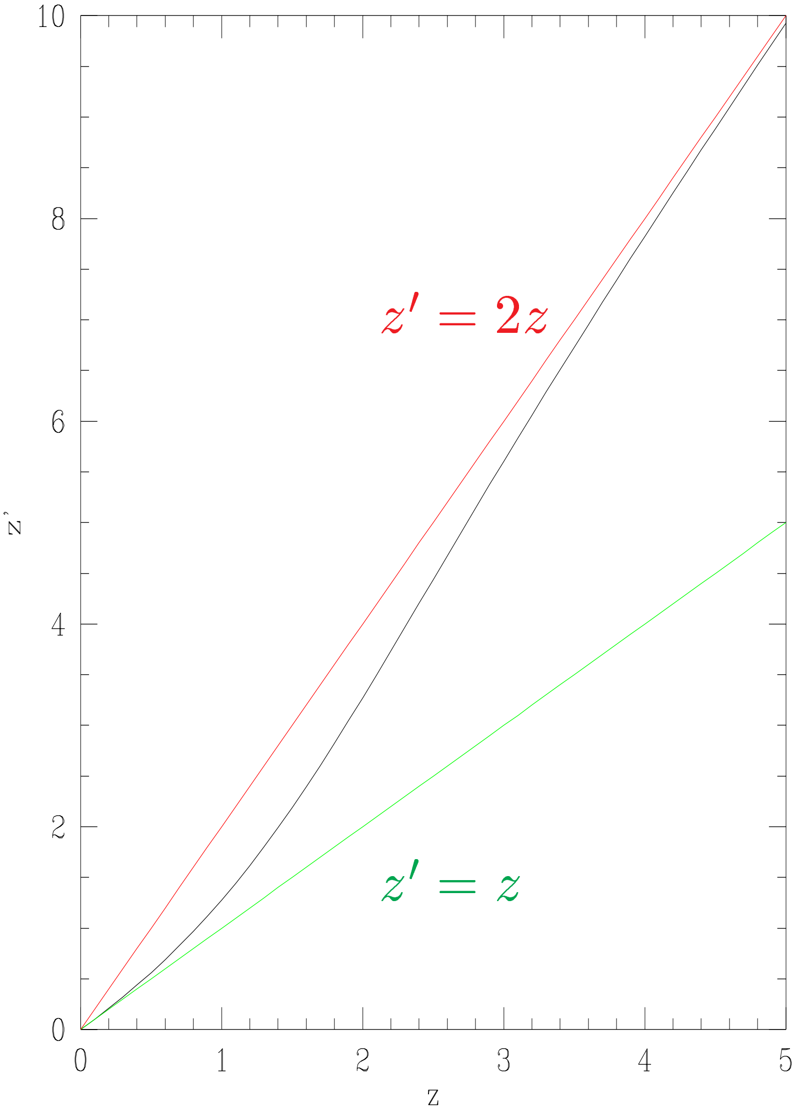
$- - -$ 2'nd order fit polynomial in $(a/L)^2$

i.e Symanzik approach to continuum limit

Approach to continuum limit of step scaling function in 2-d O(∞) σ-model



Step scaling function in 2-d O(∞) σ-model



Step-scaling function data in the O(3) σ-model

s	u	$\sigma(s, u)$
2	0.7383	0.8166(9)
2	0.8166	0.9176(8)
2	0.9176	1.0507(17)
2	1.0595	1.2641(20)
512/36	1.2680	7.904(18)

Contact with Low Energy Scale

At largest coupling volume is so big that

$M(L) \sim$ infinite volume mass gap m

Convert above table to one relating $\bar{g}^2(L)$ to mL

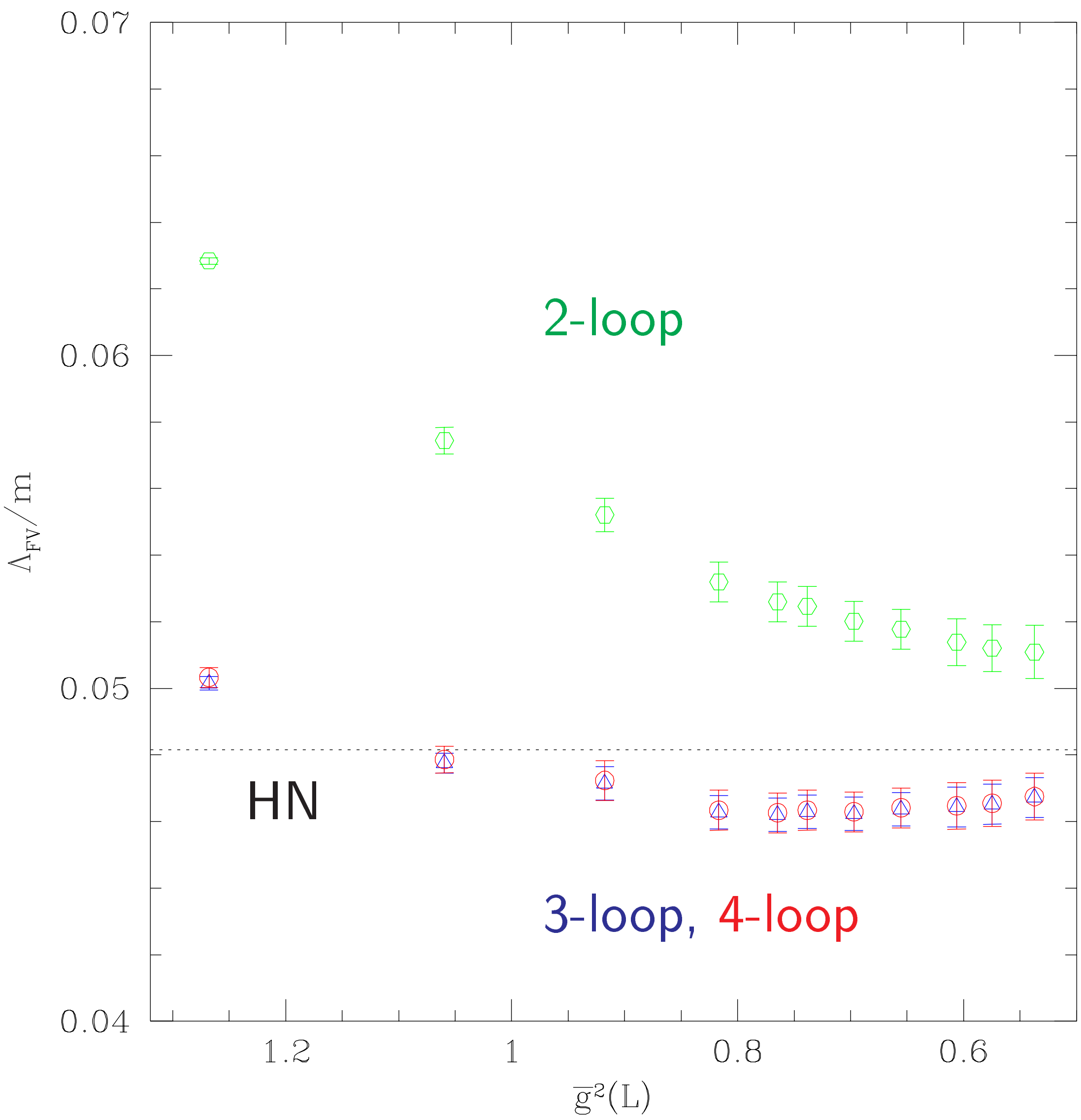
e.g. $\bar{g}^2(L) = 1.2680$ when $mL = 7.904 * 36/512 = 0.5557$

Determinations of $m/\Lambda_{\overline{\mathrm{MS}}}$ in the σ-model

Date	Authors	$n=3$	general N	Method
'82 '85	Lüscher Floratos, Petcher	1.7 ± 0.4 2.1 ± 0.2		
'82 '83 '90	Fox et al. Berg et al. Hasenfratz, Niedermayer	~ 3.7 ~ 1.3 3.4 ± 0.1		Lattice MC (SA) Lattice MC (IA)
'85 '87	Müller et al. Biscari et al. Lehmann		$1+$ $(\ln 8 + \gamma - 1)\frac{1}{N}$ $+\cdots$	$\frac{1}{N}$-Expansion
'85	Gliozzi	$2.71\cdots$	$\exp[1/(N-2)]$	Ham. Form. $+$ OPE (wrong!)
'81	Iwasaki	$8/e =$ $2.94\cdots$		Instantons ? !
'90	Hasenfratz, Maggiore, Niedermayer	$8/e$	$\frac{(8/e)^{1/(N-2)}}{\Gamma[1+1/(N-2)]}$	TBetheA $+$ S-Matrix

m/Λ in 2-d O(3) σ-model obtained
from finite size scaling analysis

Lüscher, Weisz and Wolff (92); Shin (96)



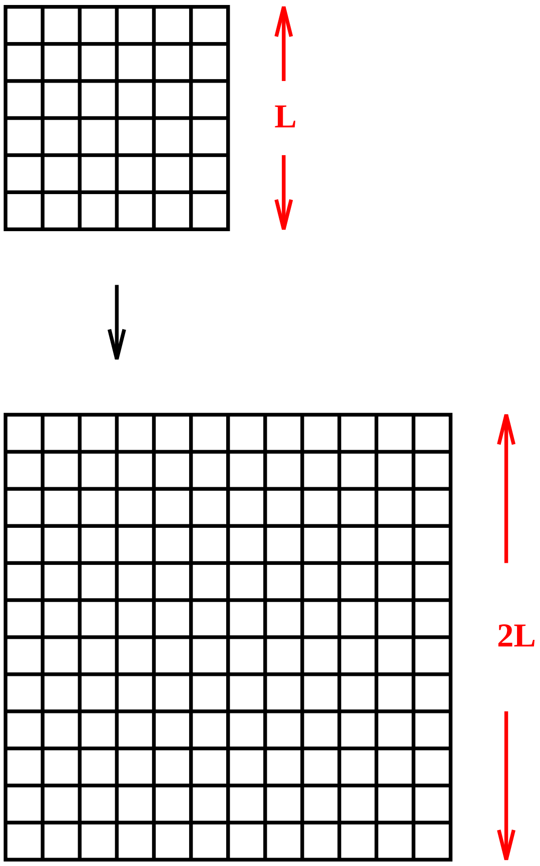
CONCLUSIONS

- Conventional wisdom seems in good shape in the 2-d O(n) non-linear sigma-model

- Recent work of Balog and Niedermaier give presently best evidence for the existence of an asymptotically free theory with mass gap

- Recursive finite size scaling technique for accurate computation of a running coupling looks promising

Computation of the scale evolution of $\bar{g}^2(L)$



choose lattice size L/a and
bare coupling g_0

compute $\bar{g}^2(L) \rightarrow$ result

keep g_0 fixed

compute $\bar{g}^2(2L)$

Chose g_0 for bigger value of L/a giving same result
Repeat....

O(3) 2d σ-model n-particle spectral functions (Isospin 0)

J. Balog and M. Niedermaier ('96)