Problem Set - Theory of General Relativity - Prof. Pierre van Baal

References to Foster & Nightingale are indicated with FN

1) General connections, symmetry, and local inertial frames

In general, a connection on a Riemannian manifold defines how to parallel transport vectors from point to point. Let P be a point having coordinates x^a, and let Q be a neighbouring point having coordinates $x^a + \delta x^a$.

Let λ^a be the components of a vector at P. Transporting this vector parallel to the point Q gives a vector having the components $\lambda^a - \Gamma^a_{bc}(P)\lambda^b\delta x^c$. Parallel transport along a curve $x^a(u)$ defines a covariant derivative in the following way

$$\frac{D\lambda^a(u)}{Du} = \frac{d\lambda^a(u)}{du} + \Gamma^a_{bc}(x(u))\lambda^b(u)\frac{dx^c(u)}{du}.$$

Essential for the principle of equivalence is the existence of a local inertial frame in which (for a general euclidean space-time $\eta_{ab} = \delta_{ab}$)

- $g_{ab} = \eta_{ab}$, where $\eta_{ab} \equiv \mathrm{diag}(1, -1, -1, -1)$.
- parallel displacement is trivial.

In this exercise we will verify that, given Γ^a_{bc}, such a local inertial frame can always be found, provided that $\Gamma^a_{bc} = \Gamma^a_{cb}$.

a) If we define Γ^a_{bc} in terms of the metric g_{ab}, the symmetry is evident. This gives a special connection, the Levi-Civita connection. The Γ^a_{bc} of the Levi-Civita connection are called the Christoffel symbols. Consider a coordinate transformation $x^a \to x'^a(x) \equiv x^{a'}$. Demand that the covariant derivative transforms as a vector, and deduce from this property how the coefficients Γ^a_{bc} of a general connection transform. Show that when $\Gamma^a_{bc} \neq \Gamma^a_{cb}$ there does not exist a local inertial frame.

b) Show now, using the transformation behaviour found in part a), that for every symmetric connection, in each point of space-time a transformation to a frame $x^{a'}$ can be found such that in this point $\Gamma^{a'}_{b'c'} = 0$.
Hint: choose coordinates x^a in which the point is given by $x = 0$, and consider the transformation $x^a \to x^a + \frac{1}{2}A^a_{bc}x^bx^c$, where A^a_{bc} is a constant.

2) From the principle of equivalence towards geometry

Starting from the principle of equivalence, or in other words the existence of local inertial frames, one can derive parallel transport, covariant derivatives, and geodesic equations in an alternative way. Throughout this exercise $x^{a'}$ denotes the local inertial frame. (*Hint*: use exercise 1b)

a) Derive from the requirement that in the inertial frame parallel transport is given by $d\lambda^{a'}/ds = 0$, the equation for parallel transport in the original frame.

b) Find the equation (FN.2.53) for the covariant derivative in the original frame. Note that in the original frame $\lambda^{a'}_{;b'} = \lambda^{a'}_{,b'}$.

c) Derive the geodesic equation using

$$\frac{d^2x^{a'}}{ds^2} = 0.$$

3) Torsion as the antisymmetric part of a connection

We use the general definition of parallel transport as in exercise 1.

a) We demand that a covariant derivative leaves invariant the length of a vector and the angle between two vectors. Show that demanding that (see also §FN.2.2 en FN.2.3.)

$$\frac{d(\lambda^a \mu_a)}{du} = \frac{D(\lambda^a \mu_a)}{Du} = \frac{D\lambda^a}{Du}\mu_a + \lambda^a \frac{D\mu_a}{Du},$$

for arbitrary vectors $\lambda^a(u)$ and $\mu_a(u)$ defined along a curve $x^a(u)$, implies that

$$\frac{D\mu_a(u)}{Du} = \frac{d\mu_a(u)}{du} - \Gamma^b_{ac}(x(u))\mu_b(u)\,\frac{dx^c(u)}{du},$$

without using $\Gamma^b_{ac} = \Gamma^b_{ca}$.

b) $T^a_{bc} \equiv \frac{1}{2}(\Gamma^a_{bc} - \Gamma^a_{cb})$ is called the torsion tensor. Show that the torsion tensor transforms as a tensor of type (1,2) using the transformation found for a general connection in exercise 1a. For any connection it holds that $\Gamma^a_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}) + T^a_{bc}$. How does it follow from this that the torsion tensor is of type (1,2)?

c) Let O be a point in curved space-time, with P and Q two other points that are infinitesimally close to O. Define $\vec{OP} \equiv \delta_1 x^a$ and $\vec{OQ} \equiv \delta_2 x^a$ to be the two vectors at O tangent to the curves OP and OQ respectively. The curves OP and OQ are the shortest curves between O on the one hand and P and Q respectively on the other hand. We now apply parallel transport to the vector $\vec{OP}$ along the curve OQ (which for an infinitesimal displacement can be seen to coincide with $\vec{OQ}$). This gives a vector $\vec{OP}'$ in the point Q, having as its endpoint P'. Similarly we apply parallel transport to the vector $\vec{OQ}$ along the curve OP. This gives a vector $\vec{OQ}'$ in the point P, having as its endpoint Q'. Draw this "parallelogram". Show that, to second order in $\delta_1 x^a$ and $\delta_2 x^a$, $P' = Q'$ if and only if the connection is symmetric, $\Gamma^a_{bc} = \Gamma^a_{cb}$ (i.e. $T^a_{bc} = 0$, this is called torsion free).

4) Physical application of the principle of equivalence

We now give an alternative derivation of $g_{00} = 1 + 2V/c^2$ in a weak gravitational field using the principle of equivalence. Let a source of light move around an observer at a constant angular speed ω. What is the change in wavelength for this observer? Write the change in wavelength in terms of the centrifugal potential $V_c = -\frac{1}{2}\omega^2 r^2$. Argue why it follows from the principle of equivalence that $g_{00} = 1 + 2V/c^2$ in a gravitational field.

5) Geodesics, connections, curvature

To get more familiar to curved spaces and geodesics we now consider the two dimensional sphere and cone embedded in $\mathbf{R}^3$ with $ds^2 = dx^2 + dy^2 + dz^2$.

sphere: $x = a\sin\theta\cos\phi,\ y = a\sin\theta\sin\phi,\ z = a\cos\theta$

cone: $x = r\cos\phi,\ y = r\sin\phi,\ z = r$.

The following questions are to be answered for both the sphere and the cone. (*Hint*: see example FN.2.2.1 and problem FN.2.3).

a) What is the metric tensor?

b) Calculate the Christoffel symbols Γ^a_{bc}.

c) Calculate the *curvature tensor*

$$R^a_{bcd} \equiv \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed}.$$

Furthermore, calculate $R_{bd} \equiv R^a_{bad}$ and $R \equiv R^a_a$. Give an interpretation of the results.

d) What are the geodesic equations? Determine the most general solutions.
Hint for the sphere: Show that the angular momentum is conserved, and use this to determine the geodesics.
Hint for the cone: If you cut open the cone, what are the geodesics? Does this agree with what you found for the curvature?

6) Holonomy on the sphere

In this exercise we will study how the orientation of a vector changes when parallel transported along a *closed* curve in a curved space. See example FN.3.3.1.

a) What are the equations for parallel transport on the sphere? (For simplicity, take the radius $r = 1$.)

b) Start with the vector $\vec{\lambda}(0) = \vec{e}_\theta$, *i.e.* the unit vector in the θ-direction. How does $\vec{\lambda}(0)$ change if it is transported along a curve with coordinates $(\theta(t), \phi(t)) = (\pi/2, t)$ with $0 \le t \le 2\pi$?

c) The same question for the curve γ consisting out of 4 pieces,

$$\gamma = \gamma_4 \circ \gamma_3 \circ \gamma_2 \circ \gamma_1,$$

where $\xi \circ \gamma$ is the curve given by first moving along γ and then along ξ. The different pieces are given by

$$\begin{array}{rclll} \gamma_1(t) & = & (\pi/2, t) & \text{for} & 0 \le t \le t_1, \\ \gamma_2(t) & = & (\pi/2 - t, t_1) & \text{for} & 0 \le t \le t_2, \\ \gamma_3(t) & = & (\pi/2 - t_2, t_1 - t) & \text{for} & 0 \le t \le t_1, \\ \gamma_4(t) & = & (\pi/2 - t_2 + t, 0) & \text{for} & 0 \le t \le t_2, \end{array}$$

with $0 \le t_1 \le 2\pi$, $0 \le t_2 \le \pi/2$.

d) Show that in this case the angle $\Delta\varphi$ by which $\vec{\lambda}(0)$ has turned equals

$$\Delta\varphi = \tfrac{1}{2} \int_A R,$$

where A is the surface enclosed by the curve γ.

e) What dou you expect for the angle with which $\vec{\lambda}(0)$ turned, after this vector has been transported along an *arbitrary* closed curve?

f) Show that the norm of $\vec{\lambda}(t)$ is a constant.

g) How does an *arbitrary* vector $\lambda^\mu(0)$ change, when transported along a circle in the $\theta = \pi/2$ plane for a space with the Schwarzschild metric. What is the essential difference with the example of the sphere?

h) Verify that also in this case the norm of $\lambda^\mu(t)$ is conserved.

7) Geodesic deviation

a) Study §FN.3.4 and make exercise FN.3.4.1.

b) The trajectories of two particles in the gravitational field of a massive object with mass M are given by $\vec{x}(t)$ en $\vec{x}(t)+\vec{\xi}(t)$. Calculate $d^2\vec{\xi}/dt^2$ up to first order in $\vec{\xi}$, using the *classical Newtonian* equation. Consider the classical limit $v \ll c$ for equation (FN.3.35) and show that

$$R^i_{0j0} = \frac{GM}{c^2|\vec{x}|^5}(3x^i x_j + \delta^i_j |\vec{x}|^2).$$

This is an indication that the space in the vicinity of earth is indeed curved.

8) Variational principle

In this exercise we will derive Einstein's equation using variational calculus. The variational principle amounts to finding the extrema of a so-called action functional, e.g.

$$S[x(t)] = \int_{t_1}^{t_2} dt\, L(x(t), \dot{x}(t)),$$

where $L(x(t), \dot{x}(t))$ is called the *Lagrangian*. As we have seen in §FN.2.1 the first order variation of S under the variation $x(t) \to x(t) + \delta x(t)$, with $\delta x(t_1) = \delta x(t_2) = 0$, is given by

$$\delta S = \int_{t_1}^{t_2} dt \left\{ \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right\} \delta x(t).$$

So the variational principle, $\delta S = 0$, leads to the Euler-Lagrange equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}.$$

These give exactly the geodesic equations if we choose $L = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$. In the solutions $x(t)$ of the geodesic equations, t is called an *affine* parameter.

a) Make exercise FN.2.1.1. Explain the name affine parameter. For massive particles we can choose the path length s as the affine parameter, for photons this is not possible. Take an arbitrarily parametrised path $x^\mu(u)$ for a photon. What would be an obvious choice for the affine parameter such a photon path?

b) Make exercise FN.2.1.2. Show that $L = (|g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu|)^{\frac{1}{2}}$ also gives the geodesic equations. So a geodesic is that curve between two points for which the length is an extremum. There can be more than one extremum (give an example in case of the sphere). For a space with a positive definite metric there always is a shortest geodesic (again not always unique, give an example in case of the sphere).

Having seen how the geodesic equations can be derived from a variational principle, we want to do the same for the Einstein equations. The geodesics are parametrised by one parameter, the Einstein equations describe *fields* and are thus parametrised by the x^μ. We therefore have to slightly adapt the variational calculus. We write the following action functional:

$$S[g^{\mu\nu}] = \int d^4x\, \mathcal{L}[g^{\mu\nu}(x^\alpha), g^{\mu\nu}(x^\alpha)_{,\rho}, g^{\mu\nu}(x^\alpha)_{,\rho,\sigma}] \quad \text{met} \quad \mathcal{L} = \sqrt{-\det g_{\mu\nu}}\, R. \qquad (1)$$

Actions for (relativistic) *field theories* like gravitation are always integrals over the whole of space-time. The action can always be written as $S = \int dt\, L$ by choosing coordinates and defining $L = \int d^3x\, \mathcal{L}$. L is called the *Lagrangian*, $\mathcal{L}$ therefore is a *Lagrangian-density*.

c) Show that $d^4x\sqrt{-\det g_{\mu\nu}}$ is invariant under coordinate transformations.

Because also R is a scalar under coordinate transformations, the action given above is an invariant. This implies covariance of the Euler-Lagrange equations. Again we fix the fields on the boundaries and demand that the action is stationary under variations

d) Show that in this way we get the following Euler-Lagrange equations (the third term in the equations comes forth from the fact that the action contains second order derivatives)

$$\frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial {g^{\alpha\beta}}_{,\mu}}\right) + \frac{\partial^2}{\partial x^\mu \partial x^\nu}\left(\frac{\partial \mathcal{L}}{\partial {g^{\alpha\beta}}_{,\mu,\nu}}\right) = 0.$$

Solving this equation takes a considerable effort. Instead of solving this equation we define an auxiliary field $H^\alpha_{\beta\gamma}(x^\mu) = H^\alpha_{\gamma\beta}(x^\mu)$ and use the Lagrangian density

$$\mathcal{L} = \sqrt{-\det g_{\mu\nu}}\, g^{\alpha\beta}(H^\lambda_{\alpha\beta,\lambda} - H^\lambda_{\alpha\lambda,\beta} + H^\kappa_{\alpha\beta}H^\lambda_{\kappa\lambda} - H^\kappa_{\alpha\lambda}H^\lambda_{\kappa\beta}). \qquad (2)$$

e) Show that the Euler-Lagrange equations are given by

$$\frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} = \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial {g^{\alpha\beta}}_{,\mu}}\right) \qquad \frac{\partial \mathcal{L}}{\partial H^\alpha_{\beta\gamma}} = \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial H^\alpha_{\beta\gamma,\mu}}\right).$$

But is this allowed? The answer to this question of course is no, unless we can show that the actions defined by (1) en (2) are equivalent.

f) Show using one of the equations found in e) that $H^\alpha_{\beta\gamma} \equiv \Gamma^\alpha_{\beta\gamma}$.

g) Show now that the other Euler-Lagrange equation found in e) in this case (and only in this case!) gives exactly Einstein's field equation.

Deriving the Einstein equations from (1) is called the *first order formalism*, the derivation from (2) is called the *second order formalism*.

9) Birkhoff's Theorem

a) In this exercise we use units such that $G = c = 1$.
Show that spherical symmetry implies that the line element can be written as

$$ds^2 = A(r,t)dt^2 - B(t,r)dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

using well chosen coordinates.
Instructions:

(i) The definition of spherical symmetry: there exist coordinates $(t, \vec{x})$ on which the rotation group $SO(3)$ acts by $(t, \vec{x}) \to (t, U\vec{x})$, $U \in SO(3)$ (3×3 orthogonal matrix with $\det U = 1$), such that ds^2 is invariant under this action of $SO(3)$ and such that the 2 dimensional surface given by $\vec{x}^2, t$ constant, is spacelike

($ds^2 < 0$). Write down all rotational invariants, to be formed out of x^μ and dx^μ, and proof that the most general form of ds^2 is given by

$$ds^2 = Adt^2 - Bdr^2 - 2Cdrdt - Dr^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where (r, θ, ϕ) are the spherical coordinates of $\vec{x}$. What can you say about A, B, C and D, and especially about their sign?

(ii) Show that we can choose $D = 1$.

(iii) Of course we want to be able to choose $C = 0$. This is not always possible globally, as can be seen from the metric of a rotating disc (angular velocity ω)

$$ds^2 = (1 - \omega^2 r^2)dt^2 - dr^2 - r^2 d\phi^2 - dz^2 - 2\omega r^2 d\phi dt.$$

Show that by choosing

$$dt' = f(x^\mu)(dt - \frac{\omega r^2}{1 - \omega^2 r^2} d\phi),$$

the metric becomes time-orthogonal. Show that for no choice of the function f the integrability conditions can be satisfied (*i.e.* that dt' is not a closed differential). Hence t' can not be defined globally. How can $t'(q)$ be defined in a unique way along a curve $x^\mu(q)$. Give an example, as simple as possible, from which it is clear that t' can not be defined globally. Show, however, that in the case of spherical symmetry one can choose $C = 0$ by properly choosing $t'(x)$.

b) Write the Einstein equations for empty space in terms of $A(r, t)$ and $B(r, t)$ and show that $A(r, t) = f(t)(1 - 2m/r)$ and $B(r, t) = (1 - 2m/r)^{-1}$, where m is a constant of integration (the <u>central</u> mass). $f(t)$ is an arbitrary function. Show that $f(t) > 0$. Finally prove that you can bring ds^2 to the Schwarzschild form. Hint: start with calculating the 01 component of the Ricci tensor.

c) Now formulate yourself a theorem about the metric for a spherically symmetric distribution of mass. (Correctly formulated this is Birkhoff's theorem!).

d) Show that as a consequence of this theorem the geometry inside a hollow, spherically symmetric distribution of mass is necessarily that of flat Minkowski space-time with, in suitably chosen coordinates, $g_{\mu\nu} = \eta_{\mu\nu}$.

e) What is the analogue of Birkhoff's theorem for the Newtonian theory of gravitation and for Maxwell's theory (consider both a <u>central</u> distribution and a cavity <u>within</u> a spherical distribution).

10) Singularities

In exercise **9** we showed (using Birkhoff's theorem) that the unique, spherically symmetric solution of the Einstein equations in empty space, is given by the Schwarzschild solution. In a similar way we can use a generalization of Birkhoff's theorem to proof that the unique, spherically symmetric solution of the Einstein equations coupled to electromagnetism is given by the *Reissner-Nordström* metric:

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) dt^2 - \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where q is proportional to the total charge Q.

a) Show that both for the Schwarzschild and in Reissner-Nordström metric only $r = 0$ is a real singularity. (A real singularity is a singularity in $R_{\mu\nu}$, $R_{\mu\nu\rho\sigma}$ or combinations thereof, that can not be removed by suitably chosen coordinate transformations. Because scalars (like R, $R_{\mu\nu}R^{\mu\nu}$ etc.) are independent of the choice of coordinates, it can be shown in a relatively simple way that $r = 0$ is a real singularity.)

b) The Reissner-Nordström metric for $q^2 < m^2$ has, unlike the Schwarzschild metric, two coordinate singularities. Which ones?

c) Show that an observer that passed the bigger of the two (r_+), always has to pass the smaller (r_-), regardless of the amount of effort and energy he spends. (Conclude this from studying the behaviour of ds^2 in the relevant region. Can you derive from this a (generous) upper limit for the proper time it takes to travel from r_+ to r_-?)

d) Can you read off from the metric if an observer necessarily will fall into the singularity once r_- has been passed?

11) Gravitational Radiation

In this exercise we will calculate the amount of energy a rotating body radiates in the form of gravitational radiation. As usual, the metric is given by $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and indices are raised and lowered using $\eta_{\mu\nu}$. Furthermore, it is assumed that the gauge conditions $h^\alpha_\alpha = 0$ and $h^\alpha_{\beta,\alpha} = 0$ hold (see §FN.5.1 and p.167). The Christoffel symbols and the curvature tensor can be written as a sum $\Gamma = \Gamma^{(1)} + \Gamma^{(2)} + \cdots$, where $\Gamma^{(i)}$ is the contribution of order i in the $h_{\mu\nu}$.

a) Show that

$$\Gamma^{(2)\mu}_{\nu\sigma} = -\tfrac{1}{2}h^{\mu\beta}(h_{\sigma\beta,\nu} + h_{\nu\beta,\sigma} - h_{\nu\sigma,\beta}).$$

b) Show that

$$R^{(2)}_{\mu\nu} = Q_{\mu,\nu} + S^\alpha_{\mu\nu,\alpha} + \frac{1}{4}h_{\alpha\beta,\mu}h^{\alpha\beta}{}_{,\nu} - \tfrac{1}{2}h^\alpha_\nu\Box^2 h_{\alpha\mu},$$

where Q and S are certain tensors you do not have to calculate explicitly.

c) Assume for the rest of this exercise that $h_{\mu\nu}$ is a plane wave solution of the Einstein equations, and that $h_{\mu\nu}$ is in the TT-gauge (see §FN.5.2). Although $G^{(1)}_{\mu\nu}(h) = 0$, when we take into account second order terms in $G_{\mu\nu}$, in general $G(h) \neq 0$, and h is *no longer* an exact solution of the Einstein equations. We could solve this by adding a second order term $\triangle h$ to h, such that $\triangle G_{\mu\nu} = \kappa T_{\mu\nu}$, with

$$T_{\mu\nu} = -\frac{1}{\kappa}G^{(2)}_{\mu\nu}.$$

Naively we would interpret $T_{\mu\nu}$ as the energy-momentum tensor of the gravitational radiation. This interpretation is problematic, because one can not describe the energy density of gravitational waves on an arbitrary small scale. Only the average over a sufficiently large region is a quantity that physically makes sense. From now on we will assume that $T_{\mu\nu}$ means the average of $T_{\mu\nu}$ over a certain region. By taking the average the total derivatives in $T_{\mu\nu}$ give much smaller contributions than the other terms. Therefore, the total derivatives in $T_{\mu\nu}$ can be set to zero for the rest of this exercise. Now show that

$$T_{\mu\nu} = \frac{c^4}{32\pi G} h_{\alpha\beta,\mu} h^{\alpha\beta}{}_{,\nu}$$

d) This result has been obtained in the TT-gauge. We can not simply use Eq. FN.5.41, since the h^{ij} given there has not yet been brought to the TT-gauge. Define

$$I^{ij} = \int \rho x^i x^j dV.$$

We want to project I^{ij} onto the component that is in the TT-gauge. To do so, we first project I^{ij} onto its traceless component, Q^{ij} (so $Q^i_i = 0$). Q is called the quadrupolemoment of the mass distribution. Give an expression for Q^{ij}.

e) Given a three vector $\vec{x} = (x^1, x^2, x^3)$, define

$$P^a_b(\vec{x}) = \delta^a_b + \frac{x^a x_b}{r^2},$$

where $r^2 = |\vec{x}|^2 = -x_i x^i$. Show that P is a projection, i.e. $P^2 = P$. What does this projection stand for?

f) Argue that Q^{ij}_{TT} is the traceless transverse component of Q^{ij}, if the gravitational radiation is emitted in the direction of $\vec{x}$, with

$$Q^{ij}_{TT} = P^i_a(\vec{x}) Q^{ab} P^j_b(\vec{x}) - \tfrac{1}{2} P_{ab}(\vec{x}) Q^{ab} P^{ij}(\vec{x})$$

g) Consider now a source of gravitational radiation placed at the origin. Show that the amount of radiated energy per unit of time through the surface S of a sphere around the origin is given by

$$\frac{dE}{dt} = \frac{G}{8\pi c^5} \int_S d\Omega\, \dddot{Q}_{TT\,ij}\, \dddot{Q}^{ij}_{TT}$$

h) Show that for the surface S of a sphere of radius R

$$\int_S d\Omega\, x^a x^b = -\eta^{ab} \frac{4}{3} \pi R^2$$

$$\int_S d\Omega\, x^a x^b x^c x^d = \frac{4}{15} \pi R^4 (\eta^{ab}\eta^{cd} + \eta^{ac}\eta^{bd} + \eta^{ad}\eta^{bc})$$

i) Show using parts g) and h) that

$$\frac{dE}{dt} = \frac{G}{5c^5} \dddot{Q}_{ij} \dddot{Q}^{ij}$$

j) Show how this leads to Eq. FN.5.44 by considering a particle that rotates around the z-axis in the xy-plane.

12) Test of the General Theory of Relativity using double pulsars.
Study the the article by J.M. Weisberg et al., Scientific American, October 1981, p.66. The orbit of one of the neutron stars of the double pulsar PSR1913+16 (a gravitationally bound system of two neutron stars with a mass of approximately 1.4 solar masses ($M_\odot$) each), has been determined accurately from timing the pulsar signals. From the slowly changing periodicity in the Doppler shift of the pulsar frequency it follows that the orbital period ($P = 27906.980895 \pm 0.000002$ s) is not entirely constant (see the figure on page 73 in: J.M. Weisberg et al., Scientific American, October 1981, 66); it has been found that $\dot{P} = -(2.427 \pm 0.026) \times 10^{-12}$.

Furthermore, from the frequency profile of the pulsar the angle between the periastron (the point of closest approach) and the line of sight can be determined. Using this, a periastron shift of $\dot{\omega} = 4.22659 \pm 0.00004$ degrees per year has been found. Furthermore, also the (high) eccentricity $e = 0.6171313 \pm 0.0000010$, as well as other orbit parameters and the gravitational redshift can be read off from this profile. By combining these data, the masses have been determined assuming $\dot{\omega}$ to be fully determined by the curvature of spacetime:

$$\begin{aligned} M_{\text{pulsar}} &= (1.442 \pm 0.003) M_\odot, \\ M_{\text{begeleider}} &= (1.386 \pm 0.003) M_\odot. \end{aligned}$$

For more background information, see J.H. Taylor e.a., Nature **227**(1979)437, or the more recent reference J.H. Taylor and J.M. Weisberg, Astrophysical Journal **345**(1989)434.

a) Proof that in the Newtonian limit the Kepler orbits of two stars (where M_1 and M_2 are of approximately the same size) are still given by Eq. FN.4.41 and FN.4.42 (where r is the distance between the two stars, i.e. in relative coordinates) if we take $M = M_1 + M_2$. Show that the total energy W is given by

$$W = -\frac{\mu G M}{2a},$$

where μ is the reduced mass $\mu^{-1} = M_1^{-1} + M_2^{-1}$ and a half the long axis of the orbit (so $a = \frac{1}{2}(r_1 + r_2) = h^2/[(1 - e^2)GM]$, see §FN.4.5). Conclude finally that in the weak field approximation Eq. FN.4.25 is also valid for the binary star system (using the fact that $2GMu^3/c^2 = 2GM_1u^3/c^2 + 2GM_2u^3/c^2$). Why do you expect there now to be higher order corrections? Finally, use the formula for the precession of the perihelion FN.4.45 ($\omega \equiv 3\varepsilon\alpha\pi$) and Kepler's law ($a^3(2\pi/P)^2 = GM$) to show that

$$\begin{aligned} \dot{\omega} &= \frac{6\pi}{1 - e^2} \left(\frac{2\pi G M}{c^3 P} \right)^{2/3} P^{-1} \\ &= 2.11 \cdot \left(\frac{M_1 + M_2}{M_\odot} \right)^{2/3} \text{ degrees per year} \end{aligned}$$

It is given that $2GM_\odot/c^2 = 2.95$ km. Because a typical neutron star has approximately 1.4 solar masses, and because the compactness of neutron stars allows us to neglect tidal effects, we may assume that the periastron shift is caused by just the relativistic effect. Check the value given for $\dot{\omega}$, using this formula, the values of the masses M_1, M_2 and the period P given above.

b) Show that

$$\frac{dW}{dt} = \frac{M_1 M_2 G}{3aP}\dot{P},$$

such that the observations imply that the binary star system is losing energy. We will now show that this can be explained by gravitational radiation. Assume first that the eccentricity can be neglected. Proof in this case that (see Eq. FN.5.44)

$$\frac{dW_{grav}}{dt}(e=0) = \frac{32\mu^2 M^3}{5a^5}\frac{G^4}{c^5},$$

and, assuming that the energy loss is due to nothing but gravitational radiation, that

$$\begin{aligned}\dot{P}(e=0) &= -2\pi\frac{96}{5}\left(\frac{2\pi G M_\odot}{Pc^3}\right)^{5/3}\frac{M_1}{M_\odot}\frac{M_2}{M_\odot}\left(\frac{M_1+M_2}{M_\odot}\right)^{-1/3} \\ &= -1.43\times 10^{-13}\frac{M_1}{M_\odot}\frac{M_2}{M_\odot}\left(\frac{M_1+M_2}{M_\odot}\right)^{-1/3}\end{aligned}$$

Note that this is a factor of 10 too small.

c) For an eccentric orbit Eq. FN.5.44 depends on time through $I(t) = \mu r^2(t) = \mu/u^2(t)$ and $\omega(t) = \dot{\phi}(t) = hr^{-2}(t) = hu^2(t)$. We will now try to approximate the average of the radiated power by taking the average of the resulting expression over one orbital period. Show that

$$\begin{aligned}\dot{W}(\mathrm{av}) &= -\frac{32h^6}{5}\frac{G}{c^5}\mu^2\langle u^8\rangle \\ &= -\frac{32h^5}{5}\frac{G}{c^5}\frac{\mu^2}{P}\int_0^{2\pi} u^6(\phi)\,d\phi \\ &= \dot{W}(e=0)\times(1+\frac{15}{2}e^2+\frac{45}{8}e^4+\frac{5}{16}e^6)(1-e^2)^{-7/2},\end{aligned}$$

so

$$\dot{P}(\mathrm{av}) = 25.1\times\dot{P}(e=0)$$

This is now two and a half times too big, but it shows clearly that the eccentricity plays an important role. Apparently our assumption is not correct. Not to despair, we now use the formula of exercise 11 which is valid in all generality.

d) Proof that $Q_{ij} = \frac{\mu r^2}{2}J_{ij}(\phi)$ (see exercise 11 for the definition of Q), where J is the following matrix:

$$\begin{pmatrix}\cos(2\phi)+\frac{1}{3} & \sin(2\phi) & 0 \\ \sin(2\phi) & -\cos(2\phi)+\frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3}\end{pmatrix}.$$

Show that with $h = r^2\dot{\phi}, u = r^{-1}$, it follows that

$$\dddot{Q} = \mu h^3(u^2(u'u''-uu''')J - 2u^3(u''+u)J'),$$

using $d/dt = hu^2 d/d\phi$ and $J''' = -4J'$.

e) Use the facts that $\mathrm{Tr}(J^2) = 8/3$, $\mathrm{Tr}(JJ') = 0$ and $\mathrm{Tr}((J')^2) = 8$ to show

$$\left\langle \mathrm{Tr}\left(\dddot{Q}^2\right)\right\rangle = \frac{1}{P}\int_0^P dt\,\mathrm{Tr}\left(\dddot{Q}^2\right) = \frac{2\pi\mu^2h^5}{P}\frac{1}{2\pi}\int_0^{2\pi} d\phi\,\left(32u^4(u+u'')^2 + \frac{8}{3}u^2(u'u''-uu''')^2\right),$$

such that

$$\dot{P}(e) = f(e)\dot{P}(e=0).$$

The correction factor

$$f(e) = \left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right)(1-e^2)^{-7/2},$$

has exactly the value, $f(e = 0.617155) = 11.86$, we are looking for.

f) Show that the curve through the data points in the figure is indeed given by the predicted value of $\dot{P}$. (The origin in this figure is at 1974.9 years.)

13) The Cosmological Constant

The Einstein equations with a cosmological constant Λ (see Eq. FN.6.21) are:

$$R^{\mu\nu} - \tfrac{1}{2}Rg^{\mu\nu} + \Lambda g^{\mu\nu} = \kappa T^{\mu\nu}$$

a) Show that

$$R^{\mu\nu} = \Lambda g^{\mu\nu} + \kappa(T^{\mu\nu} - \tfrac{1}{2}T^\lambda_\lambda g^{\mu\nu})$$

b) Find the general static, spherically symmetric solution outside of a static and spherical distribution of mass for $\Lambda \neq 0$, using that

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

c) Show that for circular orbits around the central mass the orbital time is given by

$$\triangle t = 2\pi\sqrt{\frac{r^3}{GM(r)}},$$

where $M(r) = M - c^2\Lambda r^3/3G$. Apparently Λ manifests itself as a homogeneous (background) density $\rho_0 = c^2\Lambda/4\pi G$. A rough bound for ρ_0 follows from the success of the Newtonian theory to predict planetary orbits. If for example on the basis of this one can exclude a mass defect of one earth mass ($\sim 6\times10^{24}$ kg) within the radius of the orbit of Saturn ($\sim 1.4\times10^{12}$ m) it follows that $|\rho_0| < 5\times10^{-13}$ kg m^{-3} or $|\Lambda| < 10^{-38}$ m^{-2}. By looking at the scale of a cluster of galaxies one estimates $|\Lambda| < 10^{-53}$ m^{-2}.
In the quantum theory the vacuum in principle also carries energy density. Conclude that Lorentz invariance implies that $T^{\mu\nu}_{vac} = c^2\rho_{vac}g^{\mu\nu}$ where ρ_{vac} is a constant. This energy density can not be distinguished from a real cosmological constant. Apparently there is an enormous fine-tuning that makes the effective cosmological constant $\Lambda_{eff} = \Lambda + 8\pi G\rho_{vac}/c^2 \simeq 0$. This is known as the cosmological constant problem.

d) Consider a universe without matter but with a cosmological constant Λ. Find the equations for the scale factor of the Robertson-Walker metric and solve it for arbitrary k and Λ (provided they exist). The solution with $k = 0$ is known as the De Sitter model.

e) Find the coordinate transformation that brings the De Sitter metric in the form of part b).

f) Make the metric you found Euclidean by taking time to be imaginary, and show that it is a four dimensional sphere with a radius proportional to $\Lambda^{-1/2}$. This implies that De Sitter space is maximally symmetric.