

*Contents.tex* — Revised, April 17, 2002

P.J.M. Bongaarts

Instituut Lorentz, University of Leiden  
e-mail: bongaart@lorentz.leidenuniv.nl

# TOPICS FROM 20<sup>th</sup> CENTURY PHYSICS

*An introductory course for students  
in mathematics*

Notes for a course given at the Mathematical Institute of Leiden  
University in the spring semester of 2000.

## *Preface*

Physics as we know it began in the sixteenth and seventeenth century, when in the study of natural phenomena empirical observation and mathematical modelling were for the first time systematically and successfully combined. This is exemplified in the person of Isaac Newton who laid the foundations of our picture of the physical world. He was equally great as a mathematician and as an empirical investigator and observer.

The interaction between physics and mathematics has remained of great importance. Modern theoretical physics could not exist without advanced mathematics. On the other hand many ideas in mathematics have had their origin in physics – often in a heuristic form.

The importance of the connection between mathematics and physics is no longer reflected in the curriculum of the Dutch universities. Physics students have to learn in their first and second year a great deal of standard mathematics, mainly calculus and linear algebra, but modern mathematics with its more abstract language remains strange in spirit to them, even though they sometimes pick up and learn to use methods based on it. Mathematics, on the other hand, is taught as a self-contained subject, which can be studied for its own sake, without any reference to or knowledge of physics. After a few years of training in rigorously formulated mathematics the average mathematics students will find the loose language of standard physics text books very hard to understand.

There is a strange asymmetry in the situation. Numerous books have been written explaining to physicists advanced topics from mathematics such as functional analysis, differential geometry, Lie groups and Lie algebras. Not much exists in the other direction; books on physics written for mathematicians are very rare, even though one would think that there is a need for such books.

The course for which these notes form the basis is an attempt to teach topics from modern physics to an audience of students in mathematics. It gives an introduction to the two highlights of twentieth century physics, quantum mechanics and relativity, at an advanced undergraduate level – fourth or fifth year students in the present Dutch system. The course should also be suitable for physics students with an interest in the mathematical background of physics.

Appendices A - D are added to – and should be read in parallel with – to the main text of the notes. They supply brief reviews of necessary mathematical material; this should be useful for the sake of establishing conventions and notation and to make the notes self-contained. Mathematics and physics should not be taught along historical lines, but an education in science is incomplete without the acquirement of some knowledge of its history. Appendix E provides short sketches of the lives of some of the interesting and colourful actors that have appeared in it. A selective bibliography is added.

# Contents

## I. INTRODUCTION

1. Historical introduction
  - 1.1. Classical physics: physics up to the end of the 19<sup>th</sup> century
  - 1.2. Problems of classical physics
2. Physics in the 20<sup>th</sup> century
  - 2.1. Two revolutions in physics
  - 2.2. Quantum mechanics
  - 2.3. The theory of relativity
  - 2.4. Outlook for the 21<sup>th</sup> century

## II. CLASSICAL MECHANICS

1. Historical remarks
  - 1.1. Aristotelian physics
  - 1.2. Galileo and Newton
2. Newtonian classical mechanics
  - 2.1. Newton's equations for a system of point particles
  - 2.2. Newton's equations as a system of first order equations
  - 2.3. A more intrinsic formulation
3. The Lagrangian and Hamiltonian formulation of classical mechanics
  - 3.1. Lagrangian variational problems
  - 3.2. Newton's equations as variational equations
  - 3.3. A more intrinsic formulation
  - 3.4. From Lagrangian to Hamiltonian classical mechanics
4. General Hamiltonian dynamical systems
  - 4.1. Symplectic manifolds
  - 4.2. The Poisson bracket
  - 4.3. Hamiltonian dynamical systems
  - 4.4. Darboux coordinates
  - 4.5. Hamiltonian classical mechanics

## III. QUANTUM THEORY: CHAPTERS 1 - 6

### – GENERAL PRINCIPLES

1. Historical background
  - 1.1. Introduction
  - 1.2. Atomic spectra

- 1.3. Earlier ideas
- 1.4. The emergence of quantum mechanics proper
- 1.5. Quantum theory in 20<sup>th</sup> century physics
- 2. The beginning of quantum mechanics
  - 2.1. Two different forms
  - 2.2. Unification
- 3. Modern quantum theory: general remarks
  - 3.1. A non-historic presentation
  - 3.2. Von Neumann's formulation
  - 3.3. Quantum theory and quantum mechanics
  - 3.4. The main elements
- 4. States and observables
  - 4.1. Two basic principles
  - 4.2. The third principle: interpretation of I and II
  - 4.3. The case of discrete spectrum
  - 4.4. The case of continuous spectrum
  - 4.5. Systems of observables
  - 4.6. The measurement process
- 5. Time evolution
  - 5.1. Time evolution as a 1-parameter group of operators
  - 5.2. The Schrödinger equation
  - 5.3. The Heisenberg picture
  - 5.4. Stationary states and constants of the motion
- 6. Symmetries
  - 6.1. Groups of symmetries
  - 6.2. Infinitesimal symmetries

### III. QUANTUM THEORY: CHAPTERS 7 - 9

#### - QUANTUM MECHANICS OF A SINGLE PARTICLE

- 7. Heisenberg's uncertainty relation
  - 7.1. Quantum mechanics of a particle: a summary.
  - 7.2. More on position and momentum
  - 7.3. The uncertainty relation of Heisenberg
  - 7.4. Minimal uncertainty states
  - 7.5. Examples
  - 7.6. The 3-dimensional case
- 8. Time evolution of wave functions
  - 8.1. Motion of a wave packet
  - 8.2. The free particle
  - 8.3. A particle in a box
  - 8.4. The tunnel effect

- 9. The harmonic oscillator
  - 9.1. Introduction
  - 9.2. The classical harmonic oscillator
  - 9.3. The quantum oscillator
  - 9.4. Lowering and raising operators
  - 9.5. Time evolution
  - 9.6. Coherent states
  - 9.7. Time evolution of coherent states

### III. QUANTUM THEORY: CHAPTERS 10 - 13

#### - APPLYING SYMMETRY PRINCIPLES

- 10. Translation symmetry and linear momentum
  - 10.1. Translation symmetry for a single particle
  - 10.2. Translation symmetry for a system of two particles
- 11. Rotation symmetry and angular momentum
  - 11.1. The rotation group  $SO(3)$  and its Lie algebra  $so(3)$
  - 11.2. Unitary representations in the single particle Hilbert space
  - 11.3. Rotation symmetry for a single particle
  - 11.4. The irreducible unitary representations of  $so(3)$  and  $SO(3)$
  - 11.5. The irreducible unitary representations of  $su(2)$  and  $SU(2)$
- 12. The hydrogen atom  
(*In preparation*)
- 13. Spin  
(*In preparation*)

### III. QUANTUM THEORY: CHAPTERS . . . (*In preparation*)

#### - MISCELLANEOUS TOPICS (*In preparation*)

### IV. THE SPECIAL THEORY OF RELATIVITY (*In preparation*)

### V. THE GENERAL THEORY OF RELATIVITY (*In preparation*)

### APPENDIX A: MANIFOLDS

- 1. Definition of a manifold
  - 1.1. Introductory remarks
  - 1.2. Topological manifolds
  - 1.3. Smooth manifolds

2. Tangent vectors and vector fields
    - 2.1. The tangent space at a point of a manifold
    - 2.2. Tangent vectors as point derivations
    - 2.3. Vector fields
    - 2.4. The tangent bundle
  3. Cotangent spaces and 1-forms
    - 3.1. The cotangent space at a point of a manifold
    - 3.2. 1-forms
    - 3.3. The exterior derivative
    - 3.4. The cotangent bundle
  4. General differential forms
    - 4.1. Definition of a general  $k$ -form
    - 4.2. The exterior derivative (continued)  
*(To be completed)*
  5. General tensor fields  
*(In preparation)*
  6. Riemannian manifolds
    - 6.1. Introductory remarks
    - 6.2. Definition of a Riemannian metric  
*(To be completed)*
  7. Local coordinate expressions  
*(In preparation)*
- (To be continued)*

## APPENDIX B: HILBERT SPACE

1. Introduction
2. Hilbert space: definition and some properties
  - 2.1. The definition of Hilbert space
  - 2.2. Two basic examples of Hilbert spaces
  - 2.3. Direct sum and tensor products of Hilbert spaces
  - 2.4. The inequality of Schwarz
  - 2.5. More general topological vector spaces
3. Operators in Hilbert space
  - 3.1. Operators and their domains
  - 3.2. Bounded operators
  - 3.3. Some properties of bounded operators
  - 3.4. Closed operators
  - 3.5. Hermitian adjoints of operators

4. Projections
  - 4.1. Definition and simple properties
  - 4.2. Partial ordering of projections
5. Symmetric and selfadjoint operators
  - 5.1. Introductory remarks
  - 5.2. Symmetric operators
  - 5.3. Selfadjoint operators
  - 5.4. From symmetry to selfadjointness
6. The spectral theorem
  - 6.1. The spectral theorem in finite dimensional Hilbert space
  - 6.2. The problem of formulating a spectral theorem in infinite dimension
  - 6.3. Spectral resolutions in finite dimensional space
  - 6.4. Spectral resolutions in the general case
  - 6.5. The spectral theorem for a general selfadjoint operator
  - 6.6. Examples
  - 6.7. The spectrum of a selfadjoint operator
  - 6.8. A functional calculus for selfadjoint operators
  - 6.9. Systems of commuting selfadjoint operators
7. Groups of unitary operators.
  - 7.1. Unitary operators
  - 7.2. 1-parameter groups of unitary operators
  - 7.3. General groups of unitary operators

## APPENDIX C: PROBABILITY THEORY

1. Introduction
  - 1.1. History
  - 1.2. Probability theory in physics
2. Kolmogorov's formulation of probability theory
  - 2.1. Discrete probability theory
  - 2.2. The general framework
3. Mathematical properties of distribution functions
  - 3.1. Introductory remark
  - 3.2. Distribution functions in a single variable
  - 3.3. Distribution functions in  $n$  variables
  - 3.4. Stochastic variables as a basic notion

## APPENDIX D: LIE GROUPS AND LIE ALGEBRAS

1. Groups
  - 1.1. The definition of a group
  - 1.2. Subgroups
  - 1.3. Morphisms of groups
  - 1.4. Examples of groups
  - 1.5. Quotient, products and semidirect products of groups
2. Representations of groups
  - 2.1. Definition of representation
  - 2.2. Equivalence of representations
  - 2.3. Constructing new representations from given ones
  - 2.4. Reducible and irreducible representations
  - 2.5. The central problem of group representation theory
3. Lie groups
  - 3.1. Definition of a Lie group
  - 3.2. Examples of Lie groups
  - 3.3. Infinitesimal characterization of Lie groups
4. Lie algebras
  - 4.1. Definition and basic properties
  - 4.2. Constructing new Lie algebras from given ones
  - 4.3. Examples of Lie algebras
  - 4.4. Representations of Lie algebras
5. The relation of Lie groups to Lie algebras
  - 5.1. The Lie algebra  $L(G)$  of a Lie group  $G$
  - 5.2. More about the relation between Lie groups and Lie algebras
  - 5.3. Matrix Lie groups and matrix Lie algebras

## APPENDIX E: BIOGRAPHICAL NOTES (*To be completed*)

### *SELECTED REFERENCES (In preparation)*