Topological Band Theory III

Lecture notes available at
https://www.lorentz.leidenuniv.nl/lorentzchair/

I. Topological Insulators in 3D
   - Weak vs strong
   - Topological invariants from band structure

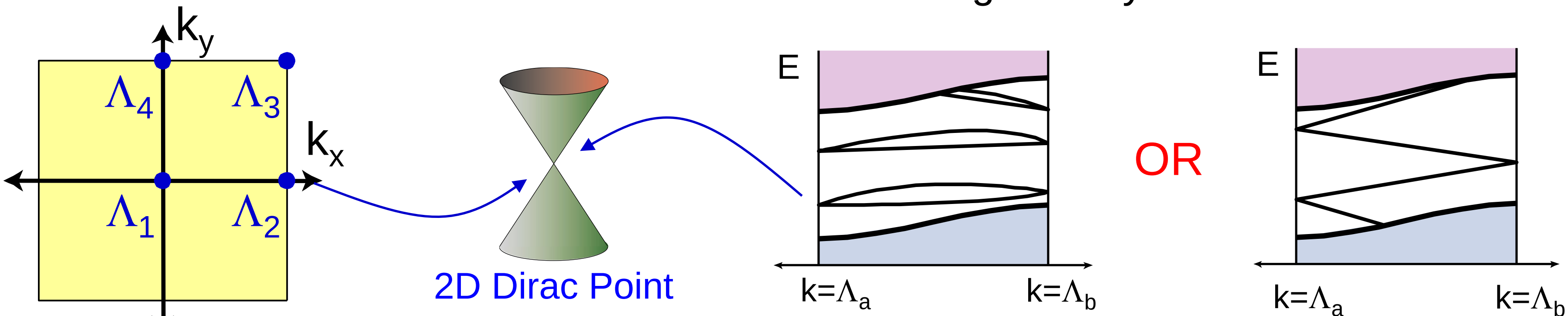
II. The surface of a topological insulator
   - Quantum Hall effect
   - θ term and topological magnetoelectric effect
   - Superconductivity
   - Surface topological order

III. Topological Superconductivity
   - 1D topological superconductor
   - Majorana chain
   - 2D topological superconductor
   - 10-fold way
   - Majorana modes and quantum information

Next Week same time, same place : Topological Mechanics

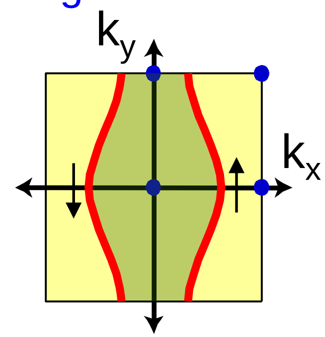
3D Topological Insulators

There are 4 surface Dirac Points due to Kramers degeneracy



Surface Brillouin Zone

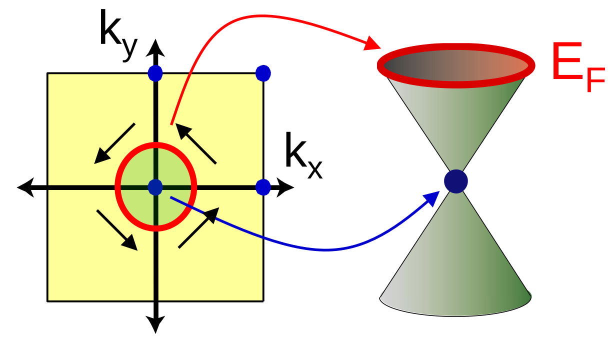
How do the Dirac points connect? Determined by 4 bulk Z_2 topological invariants ν_0 ; $(\nu_1\nu_2\nu_3)$



$\nu_0 = 0$: Weak Topological Insulator

Related to layered 2D QSHI ; $(\nu_1\nu_2\nu_3)$ ~ Miller indices
Fermi surface encloses even number of Dirac points

$\nu_0 = 1$: Strong Topological Insulator



Fermi circle encloses odd number of Dirac points

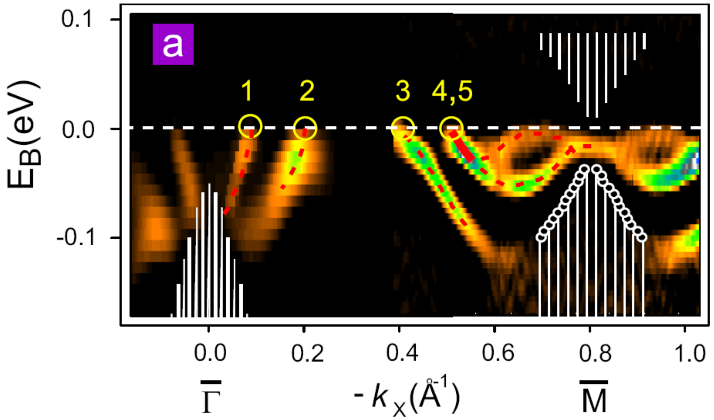
Topological Metal :

- 1/4 graphene
- Berry's phase π
- Robust to disorder: impossible to localize

$Bi_{1-x}Sb_x$

Theory: Predict $Bi_{1-x}Sb_x$ is a topological insulator by exploiting inversion symmetry of pure Bi, Sb (Fu,Kane PRL'07)

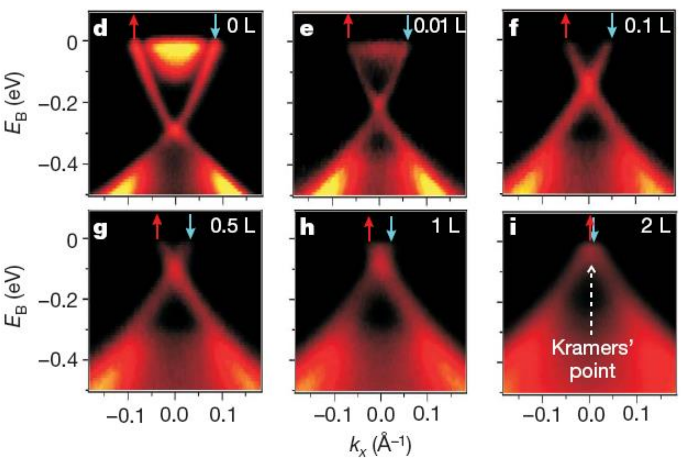
Experiment: ARPES (Hsieh et al. Nature '08)



- $Bi_{1-x}Sb_x$ is a Strong Topological Insulator $\nu_0;(\nu_1,\nu_2,\nu_3) = 1;(111)$
- 5 surface state bands cross E_F between Γ and M

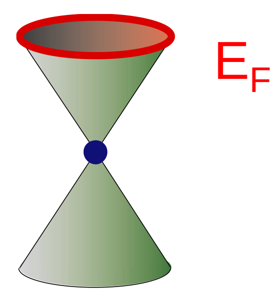
Bi_2Se_3

ARPES Experiment : Y. Xia et al., Nature Phys. (2009).
Band Theory : H. Zhang et. al, Nature Phys. (2009).



Control E_F on surface by exposing to NO_2

- $\nu_0;(\nu_1,\nu_2,\nu_3) = 1;(000)$: Band inversion at Γ
- Energy gap: $\Delta \sim .3$ eV : A room temperature topological insulator
- Simple surface state structure : Similar to graphene, except only a single Dirac point

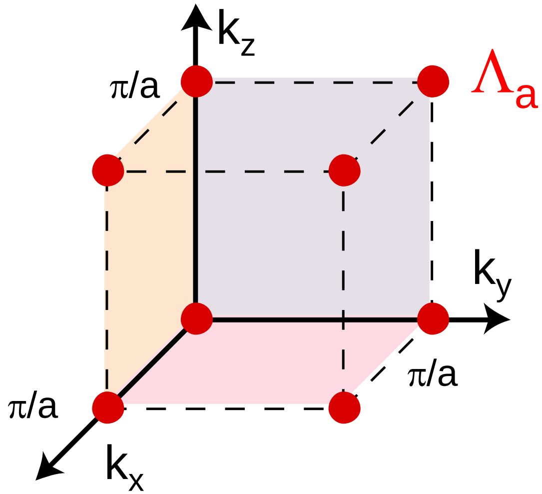


Topological Invariants in 3D

1. 2D → 3D : Time reversal invariant planes

The 2D invariant

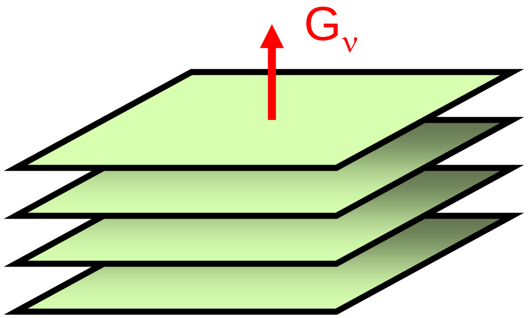
$$(-1)^{\nu} = \prod_{a=1}^{4} \delta(\Lambda_a) \qquad \delta(\Lambda_a) = \frac{\mathrm{Pf}[w(\Lambda_a)]}{\sqrt{\det[w(\Lambda_a)]}}$$



Each of the time reversal invariant planes in the 3D Brillouin zone is characterized by a 2D invariant.

Weak Topological Invariants (vector):

$$(-1)^{\nu_i} = \prod_{a=1}^{4} \delta(\Lambda_a) \Big|_{k_i=0 \text{ plane}} \qquad \mathbf{G}_\nu = \frac{2\pi}{a}(\nu_1, \nu_2, \nu_3)$$



"mod 2" reciprocal lattice vector indexes lattice planes for layered 2D QSHI

Strong Topological Invariant (scalar)

$$(-1)^{\nu_o} = \prod_{a=1}^{8} \delta(\Lambda_a)$$

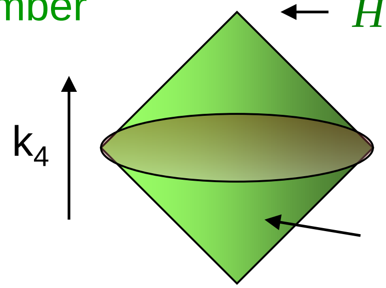
Topological Invariants in 3D

2. 4D → 3D : Dimensional Reduction

Add an extra parameter, k_4, that smoothly connects the topological insulator to a trivial insulator (while breaking time reversal symmetry)

$H(\mathbf{k},k_4)$ is characterized by its second Chern number

$$n = \frac{1}{8\pi^2}\int d^4k \, \mathrm{Tr}[\mathbf{F}\wedge\mathbf{F}]$$



n depends on how $H(\mathbf{k})$ is connected to H_0, but due to time reversal, the difference must be even.

$$\nu_0 = n \bmod 2$$

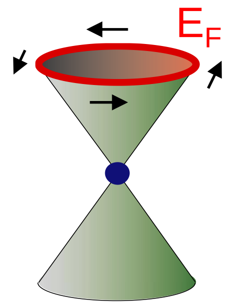
Express in terms of Chern Simons 3-form : $\mathrm{Tr}[\mathbf{F}\wedge\mathbf{F}] = dQ_3$

$$\nu_0 = \frac{1}{4\pi^2}\int d^3k \, Q_3(\mathbf{k}) \bmod 2 \qquad Q_3(\mathbf{k}) = \mathrm{Tr}[\mathbf{A}\wedge d\mathbf{A} + \frac{2}{3}\mathbf{A}\wedge\mathbf{A}\wedge\mathbf{A}]$$

Gauge invariant up to an even integer.

Unique Properties of Topological Insulator Surface States

“Half” an ordinary 2DEG ; ¼ Graphene

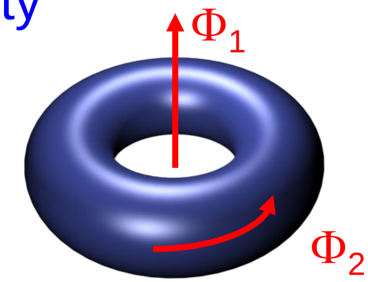


Spin polarized Fermi surface

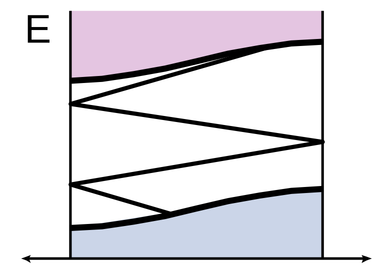
- Charge Current ~ Spin Density
- Spin Current ~ Charge Density

Robust to disorder

- π Berry’s phase
- Weak Antilocalization
- Impossible to localize



“Corbino Donut”: $\Phi_{1,2} \sim k_{x,y}$

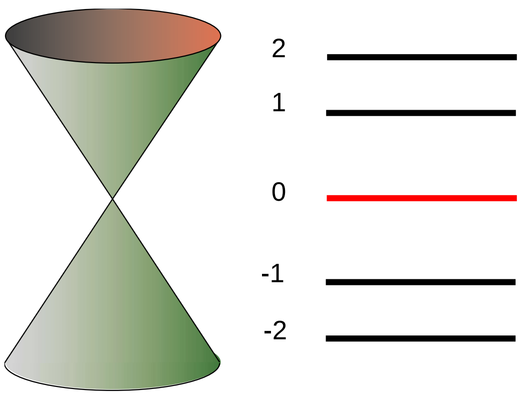


Broken symmetry (or strong interactions) leads to exotic gapped states

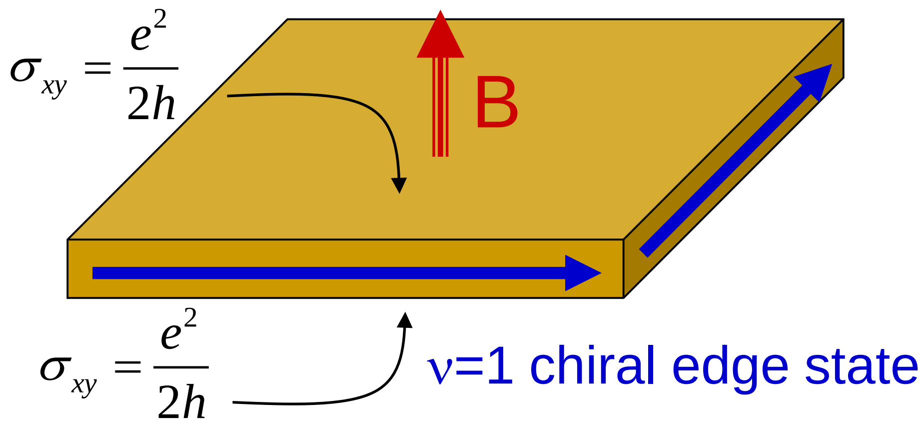
- Quantum Hall state, topological magnetoelectric effect
  Fu, Kane ’07; Qi, Hughes, Zhang ’08, Essin, Moore, Vanderbilt ‘09
- Superconducting state
  Fu, Kane ‘08
- Surface topological order: symmetry preserving gapped state
  Metlitski, Kane, Fisher ’13, Bonderson, Nayak, Qi ’13,
  Chen, Fidkowski, Vishwanath ’13, Wang, Potter, Senthil ‘14

Surface Quantum Hall Effect

Orbital QHE : E=0 Landau Level for Dirac fermions. "Fractional" IQHE



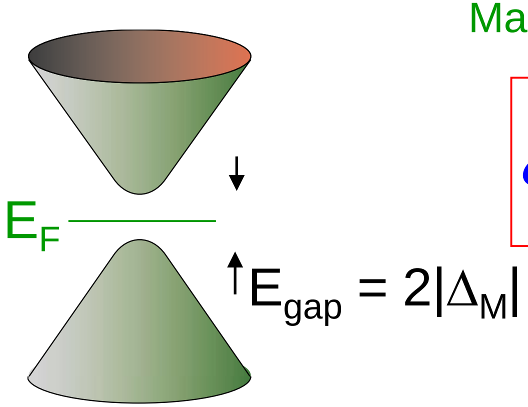
$$\sigma_{xy} = \frac{e^2}{h}\left(n + \frac{1}{2}\right)$$



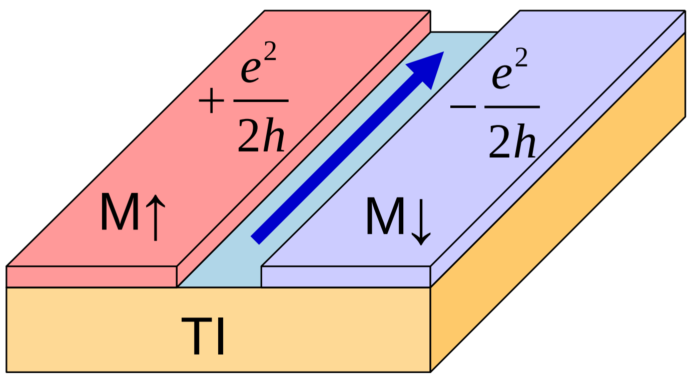
Anomalous QHE : Induce a surface gap by depositing magnetic material

$$H_0 = \psi^\dagger(-iv\vec{\sigma}\cdot\vec{\nabla} - \mu + \Delta_M \sigma_z)\psi$$

Mass due to Exchange field



$$\sigma_{xy} = \mathrm{sgn}(\Delta_M)\frac{e^2}{2h}$$

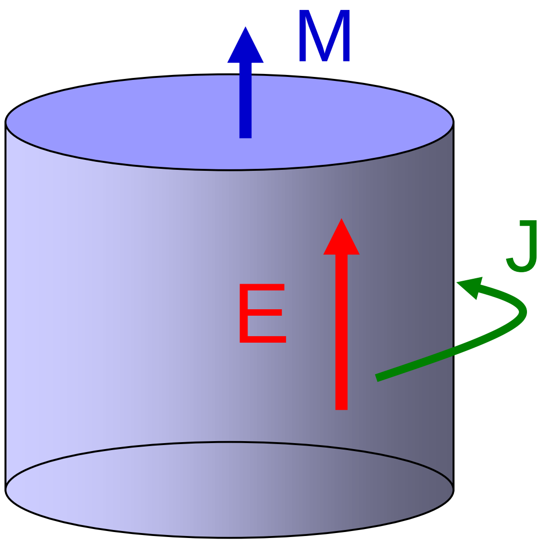


Chiral Edge State at Domain Wall : $\Delta_M \leftrightarrow -\Delta_M$

Topological Magnetoelectric Effect

Qi, Hughes, Zhang ’08; Essin, Moore, Vanderbilt ‘09

Consider a solid cylinder of TI with a magnetically gapped surface



$$J = \sigma_{xy} E = \frac{e^2}{h}\left(n + \frac{1}{2}\right) E = M$$

Magnetoelectric Polarizability

$$M = \alpha E \qquad \alpha = \frac{e^2}{h}\left(n + \frac{1}{2}\right)$$

topological “θ term”

$$\Delta L = \alpha \mathbf{E} \cdot \mathbf{B}$$

$$\alpha = \theta \frac{e^2}{2\pi h}$$

TR sym. : $\theta = 0$ or π mod 2π

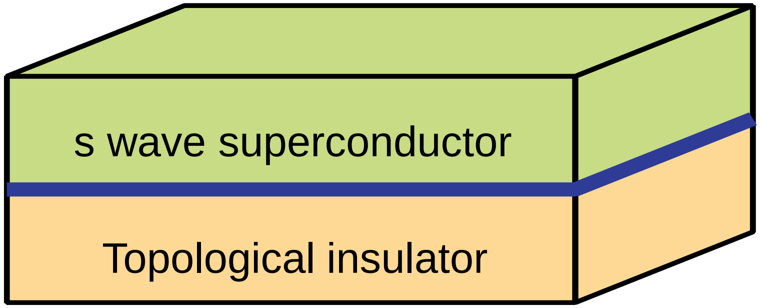
The fractional part of the magnetoelectric polarizability is determined by the bulk, and independent of the surface (provided there is a gap)
Analogous to the electric polarization, P, in 1D.

	ΔL	formula	“uncertainty quantum”	
d=1 : Polarization P	$P \cdot \mathbf{E}$	$\frac{e}{2\pi}\int_{BZ} \mathrm{Tr}[\mathbf{A}]$	e	(extra end electron)
d=3 : Magnetoelectric poliarizability α	$\alpha \mathbf{E} \cdot \mathbf{B}$	$\frac{e^2}{4\pi^2 h}\int_{BZ} \mathrm{Tr}[\mathbf{A} \wedge d\mathbf{A} + \frac{2}{3}\mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A}]$	e^2 / h	(extra surface quantum Hall layer)

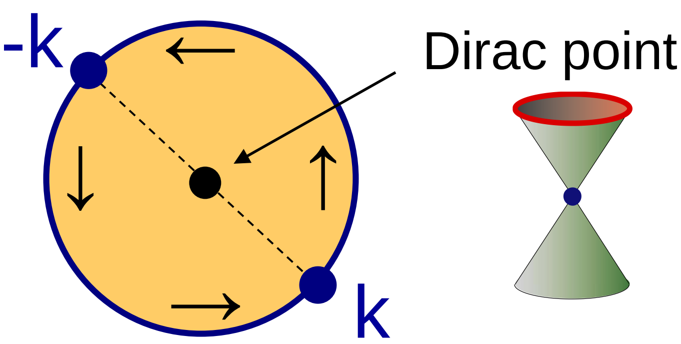
Superconducting proximity effect on a topological insulator

Proximity to superconductor introduces energy gap by breaking gauge symmetry

$$H = \psi^{\dagger}(-iv\vec{\sigma}\cdot\vec{\nabla} - \mu)\psi$$
$$+\Delta_S \psi_{\uparrow}^{\dagger}\psi_{\downarrow}^{\dagger} + \Delta_S^{*}\psi_{\downarrow}\psi_{\uparrow}$$



- Half an ordinary superconductor
- Similar to spinless p_x+ip_y superconductor, except :
  - Does not violate time reversal symmetry
  - s-wave singlet superconductivity
  - Required minus sign is provided by π Berry's phase due to Dirac Point
- Nontrivial ground state supports Majorana fermion bound states at vortices



Strong Interactions

Topological Insulator coupled to compact U(1) gauge field, A

For a compact gauge field, magnetic monopoles are excitations in the theory.
Useful diagnostic for strongly interacting theories.

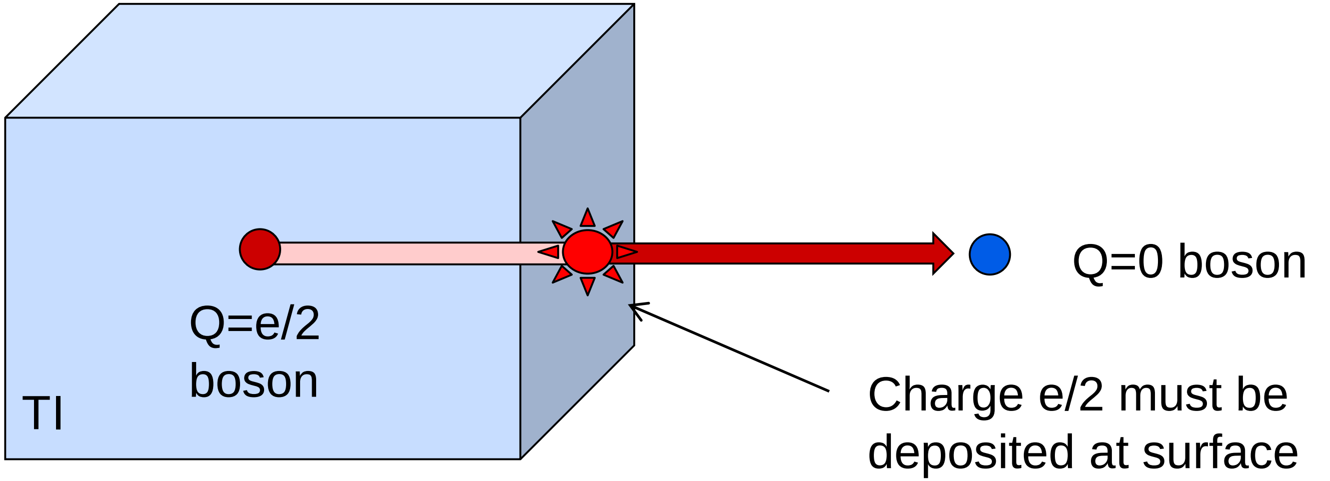
Low energy theory for A : θ term   Qi, Hughes and Zhang '08

$$S = i\theta N \qquad\qquad N = \frac{1}{32\pi^2}\int d^3x d\tau \varepsilon_{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} \in \mathbb{Z}$$

- Time reversal symmetry : $\theta = 0$ or π mod 2π. $\theta = \pi$ mod 2π for electron TI
- Witten Effect: Magnetic monopoles are charged $Q = \frac{\theta}{2\pi} e$
- Monopoles (or dyons) are bosons with half integer charge $Q = e\left(n + \frac{1}{2}\right)$

Can the surface of a TI be gapped without breaking symmetry ?

Pass monopole from inside to outside of TI :



Break T at surface:

$\sigma_{xy} = e^2/2h$: charge e/2 flows away on surface

Superconductor at surface:

Charge conservation is violated at surface

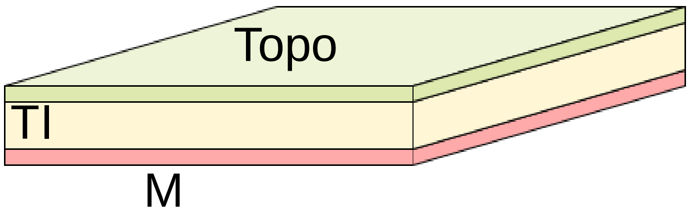
Keep U(1) and T at surface :

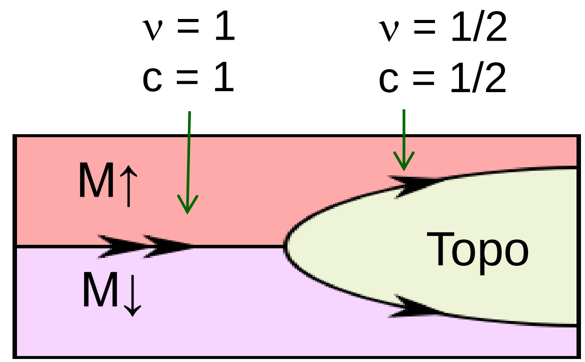
Charge e/2 stays at surface : Requires a topologically ordered surface state with e/2 quasiparticle

Requirements for a Topological Surface Phase on fermion TI

It should be impossible in 2D if symmetry is preserved, but if symmetry is broken there should be a 2D state with the same topological order

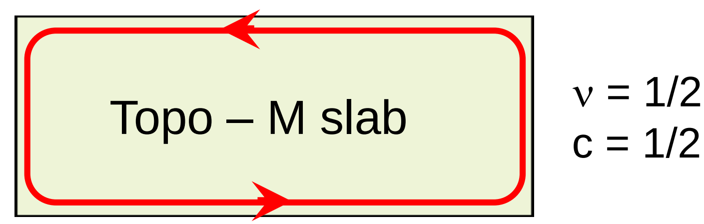
Broken T: Topo – M slab





A 2D Non-Abelian quantum Hall state with

- Hall conductance $\nu\, e^2/h$; $\nu = 1/2$
- Thermal Hall cond. $c\, \pi^2 k_B^2/6h$; $c = 1/2$



Theories of symmetry preserving gapped state:

Related to Moore-Read (Pfaffian) state of FQHE at ν=1/2

- "T-Pfaffian state" Bonderson, Nayak, Qi '13 ; Chen, Fidkowski, Vishwanath '13
- "Moore-Read/antisemion state" Metlitski, Kane, Fisher '13 ; Wang, Potter, Senthil '13

BCS Theory of Superconductivity

mean field theory : $\Psi^\dagger\Psi\Psi^\dagger\Psi \Rightarrow \langle\Psi^\dagger\Psi^\dagger\rangle\Psi\Psi = \Delta^*\Psi\Psi$

$$H = \frac{1}{2}\sum_{\mathbf{k}}\begin{pmatrix}\Psi^\dagger & \Psi\end{pmatrix} H_{BdG}\begin{pmatrix}\Psi \\ \Psi^\dagger\end{pmatrix}$$

Bogoliubov de Gennes Hamiltonian

$$H_{BdG} = \begin{pmatrix} H_0 & \Delta \\ \Delta^* & -H_0 \end{pmatrix}$$

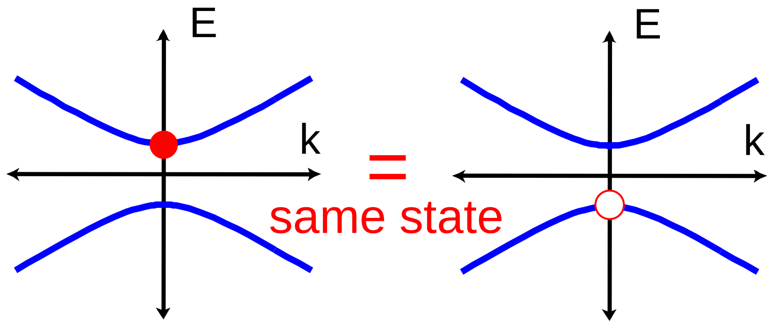
Intrinsic anti-unitary particle – hole symmetry

$$\Xi H_{BdG}\Xi^{-1} = -H_{BdG} \qquad \Xi\varphi = \tau_x\varphi^* \qquad \tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Xi^2 = +1$$

Particle – hole redundancy

$$\varphi_{-E} = \Xi\varphi_E \quad \Rightarrow \quad \gamma_E^\dagger = \gamma_{-E}$$



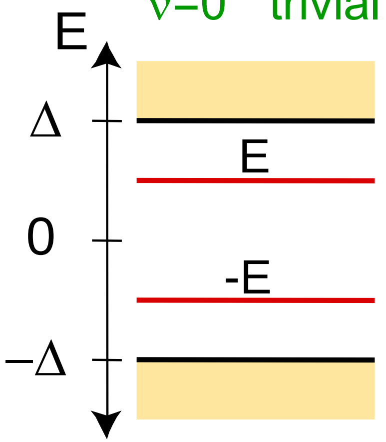
Bloch - BdG Hamiltonians satisfy $\Xi H_{BdG}(\mathbf{k})\Xi^{-1} = -H_{BdG}(-\mathbf{k})$

Topological classification problem similar to time reversal symmetry

1D $\mathbb{Z}_2$ Topological Superconductor : ν = 0,1 (Kitaev, 2000)

Bulk-Boundary correspondence : Discrete end state spectrum

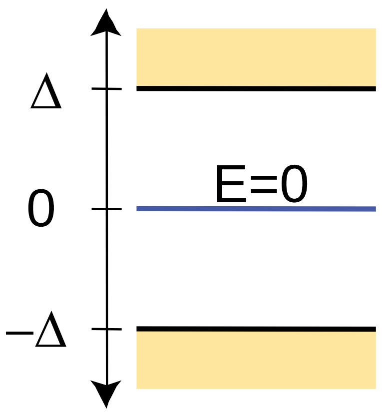
ν=0 “trivial”



Γ_E

$\Gamma_{-E} = \Gamma_E^\dagger$

ν=1 “topological”



Zero mode

$$\Gamma_{E=0}^\dagger = \Gamma_{E=0} \equiv \gamma$$

Majorana fermion bound state

Majorana Fermion : Particle = Antiparticle $\gamma = \gamma^\dagger$

Real part of a Dirac fermion :

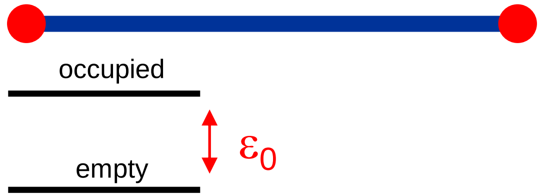
$$\begin{cases} \gamma_1 = \Psi + \Psi^\dagger & ; \quad \Psi = \gamma_1 + i\gamma_2 \\ \gamma_2 = -i(\Psi - \Psi^\dagger) & ; \quad \Psi^\dagger = \gamma_1 - i\gamma_2 \end{cases}$$

$$\gamma_i^2 = 1$$

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

“Half a state”

Two Majorana fermions define a single two level system

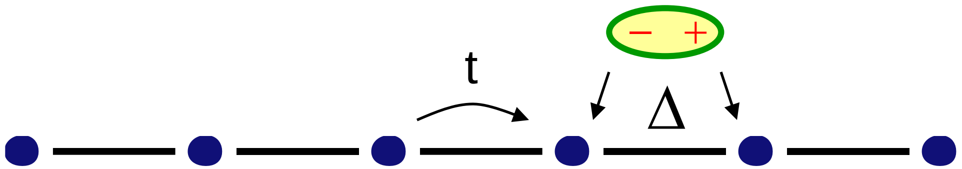


$$H = 2i\varepsilon_0\gamma_1\gamma_2 = \varepsilon_0\Psi^\dagger\Psi$$

Kitaev Model for 1D p wave superconductor

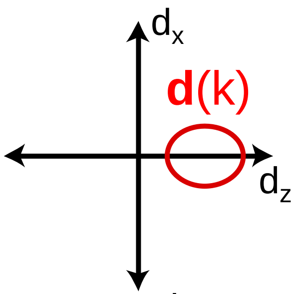
$$H - \mu N = \sum_i t(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) - \mu c_i^\dagger c_i + \Delta(c_i c_{i+1} + c_{i+1}^\dagger c_i^\dagger)$$

$$= \sum_k \begin{pmatrix} c_k^\dagger & c_{-k} \end{pmatrix} H_{BdG}(k) \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}$$

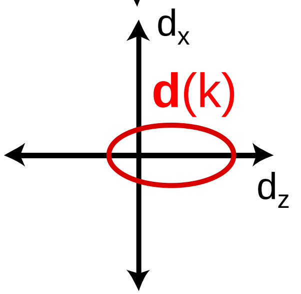


$$H_{BdG}(k) = \tau_z(2t\cos k - \mu) + \tau_x \Delta \sin k = \mathbf{d}(k)\cdot\vec{\tau}$$

$|\mu|>2t$: Strong pairing phase
trivial superconductor



$|\mu|<2t$: Weak pairing phase
topological superconductor



Similar to SSH model, except different symmetry : $(d_x, d_y, d_z)\big|_k = (-d_x, -d_y, d_z)\big|_{-k}$

Majorana Chain

$$c_i \rightarrow \gamma_{1i} + i\gamma_{2i}$$

$$\mu c_i^\dagger c_i \rightarrow 2i\mu\gamma_{1i}\gamma_{2i}$$

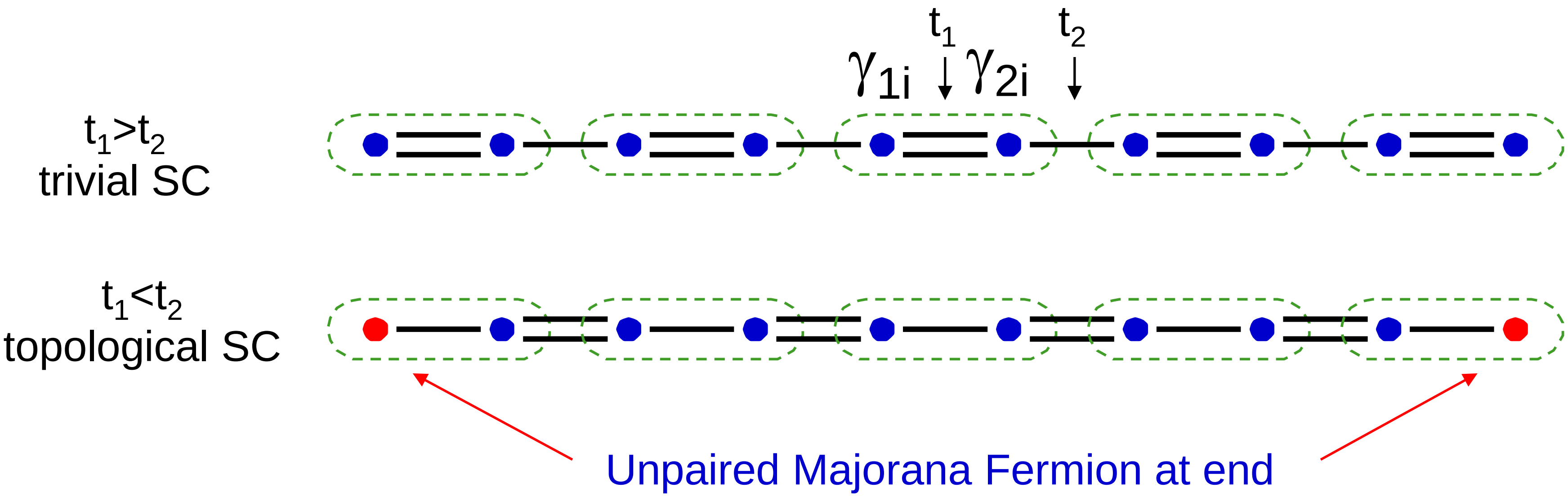
$$t\left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i\right) \rightarrow 2it\left(\gamma_{1i}\gamma_{2i+1} - \gamma_{2i}\gamma_{1i+1}\right)$$

$$\Delta\left(c_i c_{i+1} + c_{i+1}^\dagger c_i^\dagger\right) \rightarrow 2i\Delta\left(\gamma_{1i}\gamma_{2i+1} + \gamma_{2i}\gamma_{1i+1}\right)$$

$$H = 2i\sum_i t_1\gamma_{1i}\gamma_{2i} + t_2\gamma_{2i}\gamma_{1i+1}$$

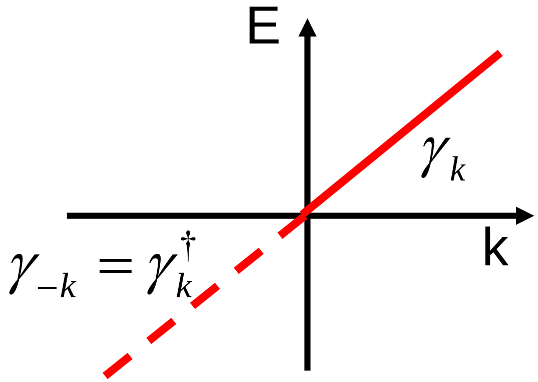
For Δ=t : nearest neighbor Majorana chain

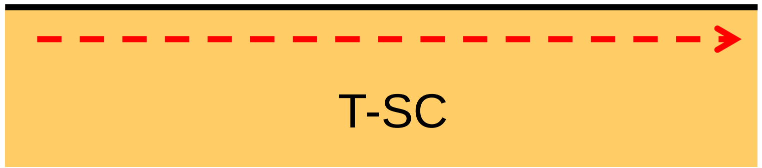
$$t_1 = \mu,\ \ t_2 = 2t$$



2D $\mathbb{Z}$ topological superconductor (broken T symmetry)

Bulk-Boundary correspondence: n = # Chiral Majorana Fermion edge states





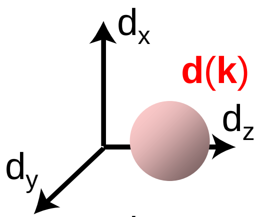
Examples

- Spinless p_x+ip_y superconductor (n=1)
- Chiral triplet p wave superconductor (eg Sr_2RuO_4) (n=2)

Read Green model : $H = \sum_{\mathbf{k}} \left(\frac{\mathbf{k}^2}{2m} - \mu \right) c_{\mathbf{k}}^\dagger c_{\mathbf{k}} + \left(\Delta(\mathbf{k}) c_{\mathbf{k}} c_{-\mathbf{k}} + c.c. \right)$ $\Delta(\mathbf{k}) = \Delta_0 \left(k_x + i k_y \right)$

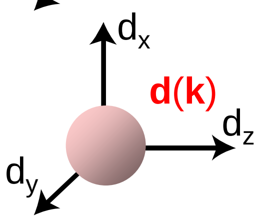
Lattice BdG model : $H_{BdG}(\mathbf{k}) = \tau_z \left(2t \left[\cos k_x + \cos k_y \right] - \mu \right) + \Delta \left(\tau_x \sin k_x + \tau_y \sin k_y \right) = \mathbf{d}(k) \cdot \vec{\tau}$

$|\mu|>4t$: Strong pairing phase
trivial superconductor



Chern number 0

$|\mu|<4t$: Weak pairing phase
topological superconductor



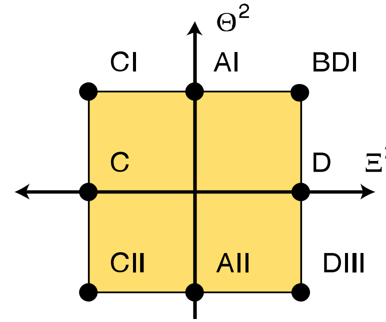
Chern number 1

Periodic Table of Topological Insulators and Superconductors

Anti-Unitary Symmetries :

- Time Reversal : $\Theta H(\mathbf{k})\Theta^{-1} = +H(-\mathbf{k})$; $\Theta^2 = \pm 1$
- Particle - Hole : $\Xi H(\mathbf{k})\Xi^{-1} = -H(-\mathbf{k})$; $\Xi^2 = \pm 1$

Unitary (chiral) symmetry : $\Pi H(\mathbf{k})\Pi^{-1} = -H(\mathbf{k})$; $\Pi \propto \Theta\Xi$



8 antiunitary symmetry classes

Symmetry				d							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	−1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	−1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	−1	−1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	−1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	−1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Altland-Zirnbauer Random Matrix Classes

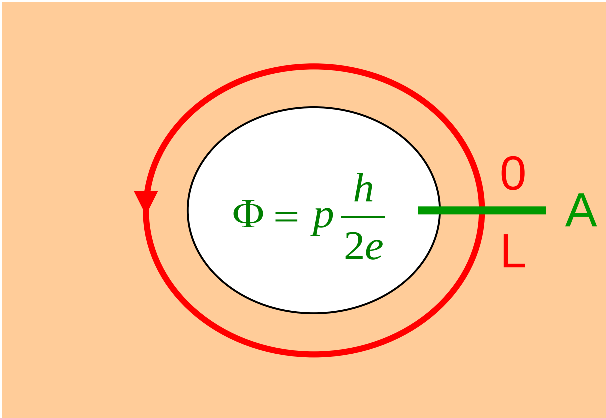
Complex K-theory (A, AIII)

Real K-theory (AI – CI)

Bott Periodicity d→d+8

Kitaev, 2008
Schnyder, Ryu, Furusaki, Ludwig 2008

Majorana zero mode at a vortex

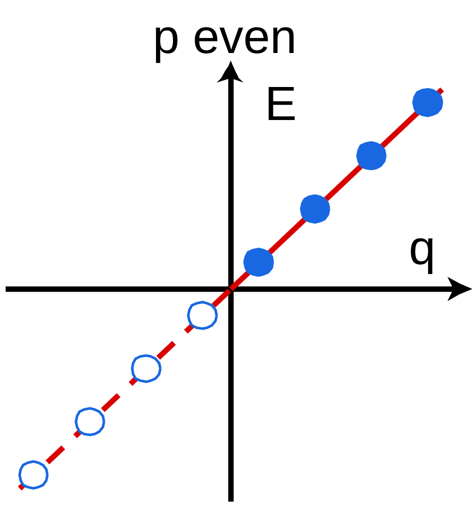


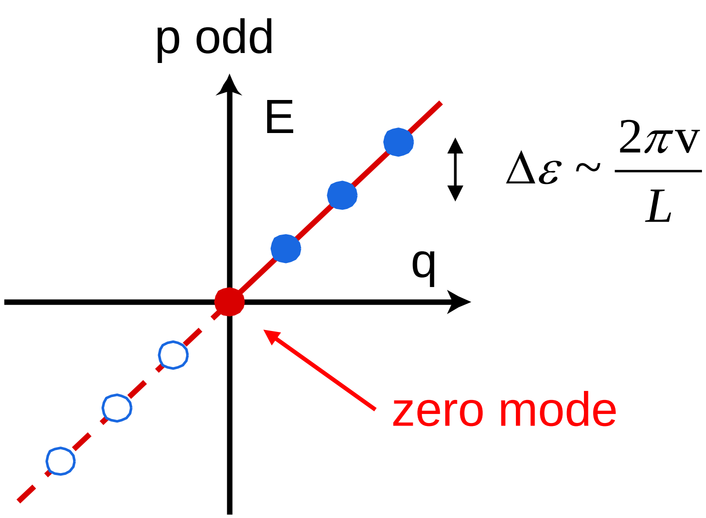
Hole in a topological superconductor threaded by flux

Boundary condition on fermion wavefunction

$$\psi(L) = (-1)^{p+1}\psi(0)$$

$$\psi(x) \propto e^{iq_m x} \quad ; \quad q_m = \frac{\pi}{L}(2m+1+p)$$

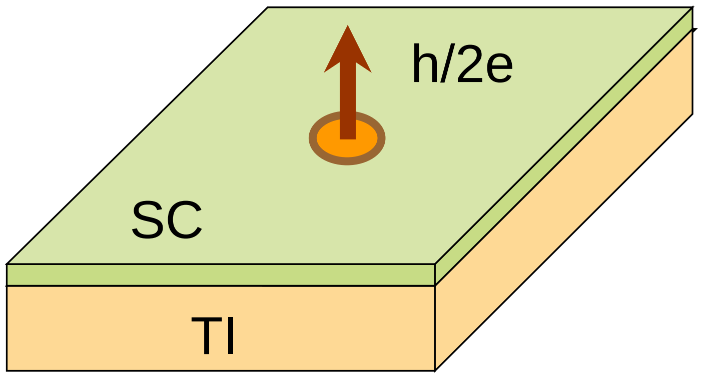


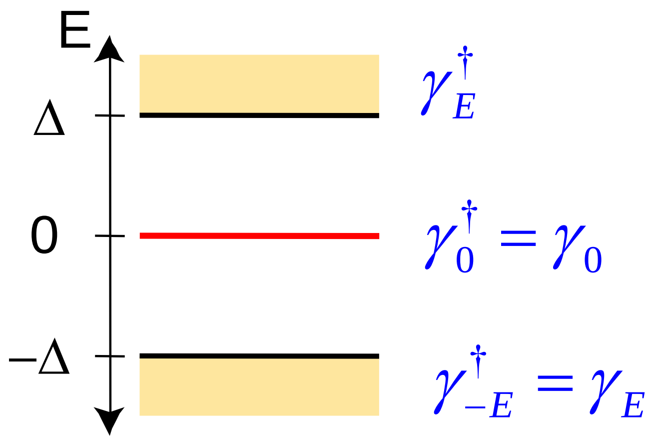


Without the hole : Caroli, de Gennes, Matricon theory ('64) $\quad \Delta\varepsilon \sim \frac{\Delta^2}{E_F}$

Majorana Bound States on Topological Insulators

1. h/2e vortex in 2D superconducting state

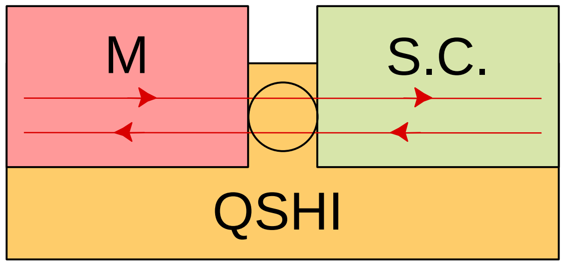




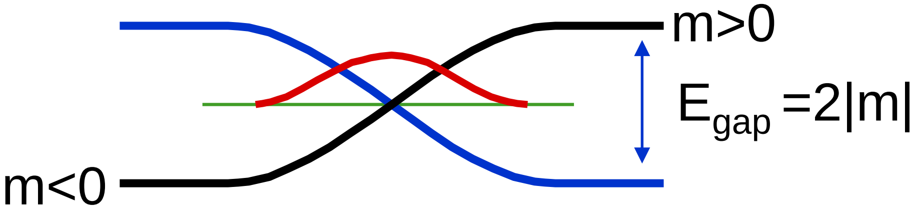
Quasiparticle Bound state at E=0

Majorana Fermion γ_0 "Half a State"

2. Superconductor-magnet interface at edge of 2D QSHI



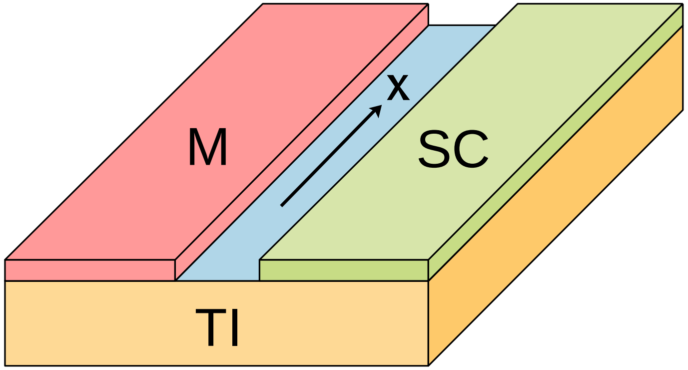
$$m = |\Delta_S| - |\Delta_M|$$

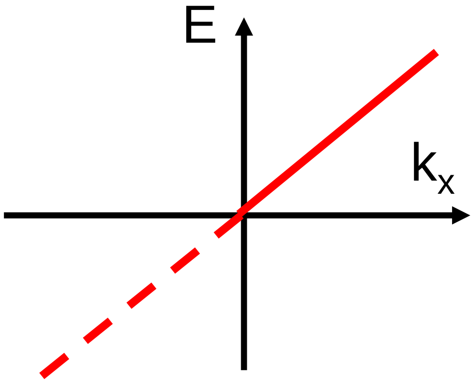


Domain wall bound state γ_0

1D Majorana Fermions on Topological Insulators

1. 1D Chiral Majorana mode at superconductor-magnet interface

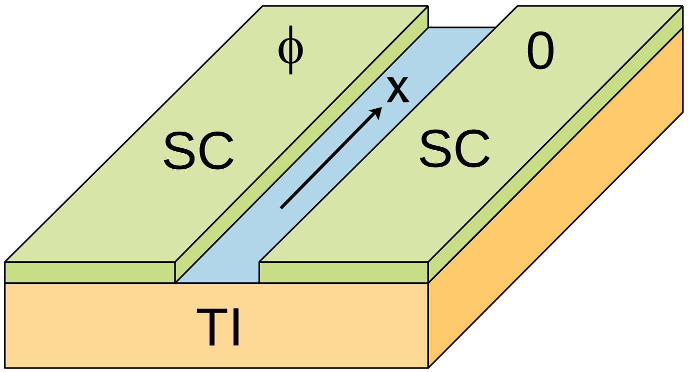


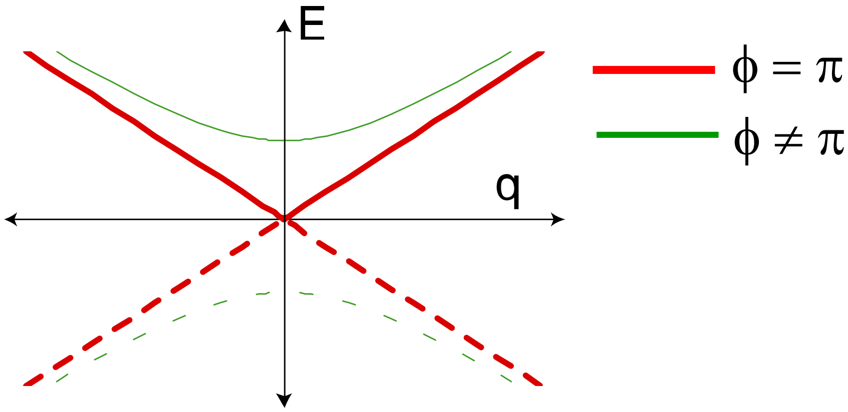


$\gamma_k = \gamma^\dagger_{-k}$: "Half" a 1D chiral Dirac fermion

$$H = -i\hbar v_F \; \gamma \partial_x \gamma$$

2. S-TI-S Josephson Junction





Gapless non-chiral Majorana fermion for phase difference $\phi = \pi$

$$H = -i\hbar v_F \; \left(\gamma_L \partial_x \gamma_L - \gamma_R \partial_x \gamma_R\right) + i\Delta \cos(\phi/2)\gamma_L \gamma_R$$

Majorana Fermions and Topological Quantum Computing

(Kitaev '03)

The degenerate states associated with Majorana zero modes define a topologically protected quantum memory

- 2 Majorana separated bound states = 1 fermion $\Psi = \gamma_1 + i\gamma_2$
  - 2 degenerate states (full/empty) = 1 qubit
- 2N separated Majoranas = N qubits
- Quantum Information is stored non locally
  - Immune from local decoherence

Braiding performs unitary operations

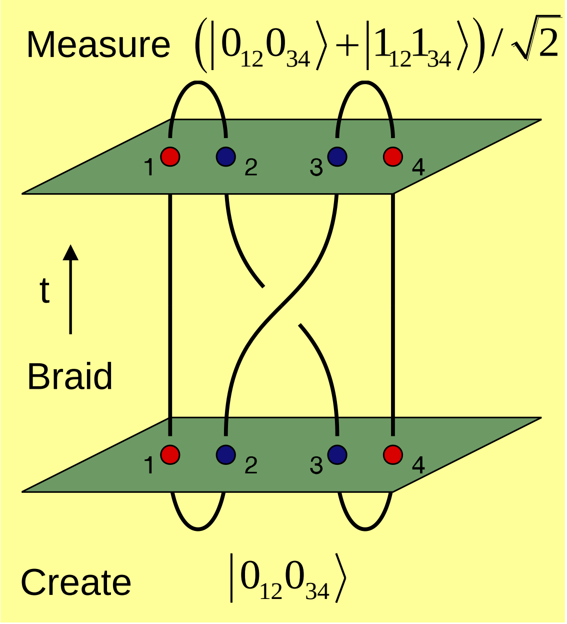
Non-Abelian statistics

Interchange rule (Ivanov 03)

$$\gamma_i \to \gamma_j$$

$$\gamma_j \to -\gamma_i$$

These operations, however, are not sufficient to make a universal quantum computer



Potential condensed matter hosts for Majorana modes

- Quasiparticles in fractional Quantum Hall effect at $\nu=5/2$ Moore Read '91

- Unconventional superconductors

  - Sr_2RuO_4 Das Sarma, Nayak, Tewari '06
  - Fermionic atoms near feshbach resonance Gurarie '05
  - $Cu_xBi_2Se_3$?

- Proximity Effect Devices using ordinary superconductors

  - Topological Insulator devices Fu, Kane '08
  - 2D Semiconductor/Magnet devices Sau, Lutchyn, Tewari, Das Sarma '09, Lee '09
  - 1D Semiconductor devices:
    eg In As quantum wires Oreg, von Oppen, Alicea, Fisher '10
    Lutchyn, Sau, Das Sarma '10
    Expt : Maurik et al. (Kouwenhoven) '12

  - 1D Ferromagnetic atomic chains on superconductors
    Expt : Nadj-Perg et al. (Yazdani) '14