The Struggles of Paul Ehrenfest

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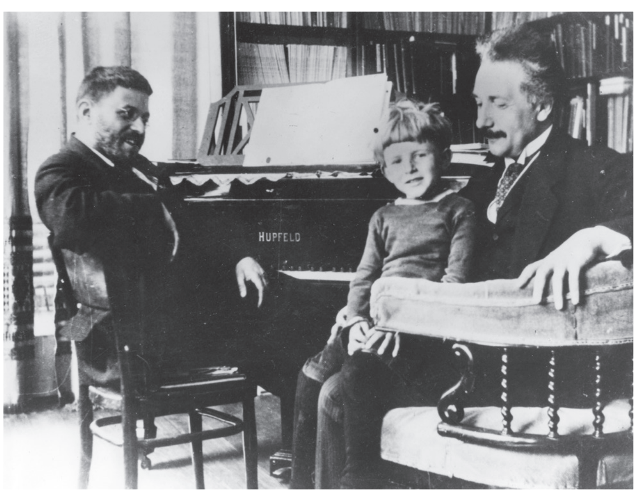
During the founding years of quantum mechanics, the Niels Bohr Institute in Copenhagen, Denmark, hosted an annual spring meeting where the hottest topics in physics were discussed. It became traditional to close the meeting with a skit that parodied the state of physics at the time. The 1932 conference coincided with the tenth anniversary of the institute's founding and needed to end on a special note.

A few years earlier, Wolfgang Pauli had suggested the existence of a particle carrying zero mass and zero electric charge that could explain the missing energy and momentum in beta decay, a type of radioactive decay of the nucleus. He called this hypothetical particle the "neutron." When in 1932 James Chadwick discovered the massive neutral particle we now call the neutron, Enrico Fermi suggested that the name of Pauli's particle be changed to *neutrino*, the "little neutral one."

At the Bohr Institute's 1932 meeting, Pauli's neutrino was still a speculative teaser, with many doubters. This offered the perfect theme for the 1932 closing skit, modeled on Goethe's *Faust*, in which Faust, who has attempted to master all knowledge and remains frustrated, makes a pact with the devil (Mephistopheles), who will do Faust's bidding in this life. With Mephistopheles' help, Faust seduces the beautiful Gretchen, but their relationship leads to tragedy. Of the brilliant Copenhagen parody, which portrayed Pauli and Paul Ehrenfest, George Gamow wrote, "The theme of this dramatic masterpiece has Pauli (Mephistopheles) trying to sell to the unbelieving Ehrenfest (*Faust*) the idea of a weightless neutrino (Gretchen)."[1]

Ehrenfest (1880–1933) was born in Austria and earned his PhD under Ludwig Boltzmann. During his career Ehrenfest made fundamental contributions to statistical mechanics and quantum theory, and was held in the highest esteem by his students and colleagues. In September 1912, on the recommendation of Arnold Sommerfeld, he was offered the prestigious chair at the University of Leiden in the Netherlands, held by the departing H.A. Lorentz.



PAUL EHRENFEST (left), Ehrenfest's son, Paul, Jr., and Albert Einstein. Ehrenfest and Einstein were close friends, the latter staying in the Ehrenfest home whenever he visited Leiden, Netherlands. Image credit: Museum Boerhaave, Leiden, courtesy of the Emilio Segrè Visual Archives.

Paul Ehrenfest always insisted on honesty in thought and action. In the preface to Ehrenfest's collected papers, H.B.G. Casimir wrote that Ehrenfest's lectures were brilliant in an unconventional way:[2]

> *He emphasized salient points rather than continuity of argumentation; the essential formulae appeared on the blackboard almost as aesthetical entities and not only as links in a chain of deductions. . . . One had very little inclination to go to sleep during Ehrenfest's lectures, but if one ever showed any tendency in that direction one was immediately and ruthlessly called to order.*

In his 1934 memorial essay for Ehrenfest, Albert Einstein recalled when they first met 22 years earlier: "We also discussed the theory of relativity, to which he responded with a certain skepticism but with the critical judgment peculiar to him. Within a few hours we were true friends—as though our dreams and aspirations were meant for each other. We remained joined in close friendship until he departed this life."[13] Whenever Einstein visited Leiden, he stayed in the Ehrenfest home.

The authors of the *Faust* parody no doubt chose Ehrenfest as Faust because of his inherent skepticism, his brilliance, his mastery of a comprehensive range of subjects, and his sterling standards of truth. Casimir recalled that "To Ehrenfest . . . discussions and arguments were an essential part of his scientific activity and the best way to clarify an obscure point."[2] Ehrenfest's skepticism of *Gretchen* was genuine.

But whether or not it was consciously recognized by the *Faust* parody playwrights, Ehrenfest's personality also fit the gloominess of the original Faust. In the 1932 skit, the opening scene begins with Faust (Ehrenfest) brooding with dissatisfaction over the vast knowledge he has mastered:

I have–alas–learned Valence Chemistry,
Theory of Groups, and the Electric Field,
And Transformation Theory as revealed
by Sophus Lie in eighteen-ninety-three.
Yet here I stand, for all my lore,
No wiser than I was before.

From the factual notes about his career mentioned above, we might assume that life for Ehrenfest was always a forward-looking adventure. Far from it. From a young age he was a melancholy figure who struggled with chronic depression. Paul was the youngest of five brothers and suffered from poor health as a child. He was often the target of anti-Semitism. His mother died when he was 10. He loathed school, and his academic performance suffered. His school experiences were sufficiently negative that, years later, he insisted that his own children be educated at home.[4]

His biographer, Martin Klein, wrote, "He was often miserable, deeply depressed, and at odds with himself and the world." When Paul was 16 his father died. Older brother Arthur convinced Paul to remain in school, and his outlook seemed to improve. Klein continues, "Paul was apparently able to work himself out of depression, which had sometimes been deep enough to make him contemplate suicide. His intellectual interests grew stronger, perhaps as a form of self-protection."[4]

In 1899 Paul enrolled in the Technische Hochschule in Vienna and attended lectures by Boltzmann on the new subject of statistical mechanics. It was under Boltzmann's influence that Paul's loathing of school was replaced by a passion for physics and mathematics. European students in those days typically migrated from one university to another to study under a variety of mentors. Starting in 1901, Ehrenfest took courses at the University of Göttingen under David Hilbert, Walther Nernst, Felix Klein, Johannes Stark, and Karl Schwarzschild. At Göttingen he met a young Russian mathematics student, Tatyana Alexeyevna Afanasyeva. At that time women were not allowed to attend meetings of the mathematics club, a rule that Ehrenfest successfully challenged after "quite a battle."[4] (His tenacity would resurface after Hitler came to power, when he found jobs for German Jews fleeing the Nazis.[5]) He returned to Vienna in 1904 and completed his doctorate that June under Boltzmann's direction, writing a dissertation on the motion of rigid bodies in fluids. His advisors respected this work, but, as was the pattern throughout his life, Ehrenfest felt it to be inadequate. He did not publish his dissertation. However, Tatyana soon joined him and they were married in Vienna that December.[4]

After finishing his PhD, Dr. Ehrenfest had difficulty securing a permanent position. Despite repeated residences in Vienna, Göttingen, and St. Petersburg, his letters of application to numerous institutions in Europe and North America proved fruitless. He published several important papers between 1904 and 1912, but his situation did not stabilize until he was offered the Leiden position in 1912. Even then, Ehrenfest continued suffering from unrealistic self-doubt.

EHRENFEST AND STATISTICAL MECHANICS

Boltzmann, Ehrenfest's mentor, was a principal founder of statistical mechanics. The Boltzmann transport equation, an inhomogenous equation of continuity, describes the evolution of the velocity distribution function for particles of an ideal gas;[7] its equilibrium solution is the Boltzmann factor, $P_n \sim \exp(-E_n/kT)$, which gives the probability for a particle to be in state n at energy E_n in an environment at absolute temperature T. The factor k converting temperature to energy is today called "Boltzmann's constant." Boltzmann thereby founded an approach to equilibrium and out-of-equilibrium thermodynamics, based on the assumption that a macroscopic system can be partitioned into a set of independent microscopic subsystems, i.e., atoms. The keystone concept in Boltzmann's approach was his celebrated "H theorem." The quantity H was Boltzmann's name for the average value of the logarithm of P_n, and the H theorem demonstrates that H in an isolated system never increases.[8] This allows the definition of entropy, which never decreases in an isolated system, to be proportional to $-H$, leading to the microscopic interpretation of entropy as a measure of disorder.

Boltzmann and Ehrenfest were kindred spirits not only in their love of statistical mechanics, but also in their melancholy dispositions. In 1906 the physical reality of atoms was still controversial. Jacob Bronowski wrote,[9]

Had anti-atomic doctrines then really won the day, our advance would certainly have been set back by decades, and perhaps a hundred years. . . . Did Boltzmann just argue? No. He lived and died that passion. In 1906, at the age of sixty-two, feeling isolated and defeated, at the very moment when atomic doctrine was going to win, he thought all was lost, and committed suicide.

Ehrenfest wrote an obituary for his mentor and friend. Boltzmann had promised the editors of *Enzyklopädie der Mathematischen Wissenschaften* an article on statistical mechanics, and the editors asked Ehrenfest to write it in Boltzmann's place. He and his wife and colleague Tatyana worked together on this project,[10] which took longer than expected but was expanded into the classic book, *The Conceptual Foundations of the Statistical Approach in Mechanics.*[11] In it they made some clarifying distinctions in Boltzmann's assumptions and simplified his proof of the H theorem.

EHRENFEST AND QUANTUM MECHANICS

Thus, one of Paul Ehrenfest's major contributions to physics was laying a groundwork in statistical mechanics that facilitated the statistical interpretation of the quantum mechanics soon to come. Important among these was his recognition of the importance of the "action" J, the closed-path line integral of momentum times displacement:

$$J \equiv \oint p \cdot dr. \tag{1}$$

The action plays an important role in the classical dynamics of periodic systems. In classical mechanics, J was shown to be an "adiabatic invariant." Adiabatic invariance occurs when a system exhibits a conservation law that would not otherwise occur, when changes are made very slowly so that there is little variation in system parameters during one cycle. For instance, if a simple pendulum's length changes, neither its period T nor its energy E remain constant. But if the change is made sufficiently slowly, then the time-average of ET is conserved.[12]

In the early days of confronting quantized oscillators, quantized radiation, and quantized atoms, with their finite energy gaps between energy levels, it was thought the system should not make transitions between states *too* easily. Ehrenfest realized that adiabatic invariants, notably J, could be the path to a quantum theory. David Bohm noted, "In fact, Ehrenfest originally argued from the adiabatic invariance of J that this was the only classical quantity that could sensibly be quantized."[13] This inspired the algorithm of setting $J = nh$ for periodic systems, the Bohr-Sommerfeld-Wilson quantization rules, with n an integer and h the quantum. Bohr's 1913 model of the hydrogen atom offers a famous example.[14] More generally, the energy of a periodic system becomes quantized through J because $p = \pm\sqrt{[2m(E—U(x))]}$. Thus came about the precursor of quantum mechanics, what we now call "old quantum theory," based on adiabatic invariance as recognized by Ehrenfest. Old quantum theory held the field for about 10 years, between the Bohr atom and the advent of quantum mechanics in terms of de Broglie wave functions and the Schrödinger equation that took over in the mid-1920s.

One of the beauties of science is that its revolutions do not result in the leaders of the old order being hanged and their books being burned. Revolutions in physics seldom discard preceding theories. Rather, the old ideas typically become special cases of the new. Indeed, as a new theory gets constructed, this "correspondence principle" offers guidance.

Let us explore one elegant display of the correspondence principle that occurs with "Ehrenfest's theorem." (See your favorite quantum mechanics textbook for reference.) The theorem shows that, in general, quantum mechanics contains Newtonian mechanics as a special case. Newton's second law says that for a net force derivable from a potential energy function $U(x)$ (here considering one-dimensional motion), a particle's momentum changes with time according to

$$-\frac{dU}{dx} = \frac{dp}{dt}\,. \tag{2}$$

Newtonian mechanics assumes the existence of a precise particle trajectory for which the instantaneous location *and* momentum can be simultaneously known, in principle, to an infinite number of decimal places. But quantum mechanics changes that picture.

For a particle moving along the x axis, quantum mechanics calculations are done in terms of a complex number $\Psi(x,t)$, called a "wave function." Its square, $\Psi^*\Psi$ (* denotes complex conjugate), is a probability distribution. The probability $P(a,b)$ of locating the particle in the interval $[a,b]$ along the x axis is

$$P(a,b) = \int_a^b \Psi^*\Psi\, dx \tag{3}$$

where $P(-\infty,+\infty) = 1$ because the particle has to be *somewhere* so long as it exists. The ensemble average value of an observable Q (represented in the formalism as an operator) follows from the rules of statistics with a continuous probability density,

$$\langle Q \rangle = \int_{-\infty}^{+\infty} \Psi^* Q \Psi\, dx. \tag{4}$$

These maneuvers describe what can be done with Ψ when we know it. The way to find Ψ, given the particle's mass m and the potential energy function whereby it interacts with the rest of the world, is to solve the Schrödinger equation,

$$\frac{-\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + U(x)\Psi = \frac{-\hbar}{i}\frac{\partial\Psi}{\partial t} \tag{5}$$

or, for brevity,

$$H\Psi = \frac{-\hbar}{i}\frac{\partial\Psi}{\partial t} \tag{6}$$

where $H = p^2/2m + U$ (the "Hamiltonian"), with

$$p\Psi = \frac{\hbar}{i}\frac{\partial\Psi}{\partial x}\,. \tag{7}$$

This statement about momentum being a derivative is equivalent to the de Broglie postulate that, corresponding to a free particle of momentum p, there exists a harmonic wave of wavenumber k, $\Psi \sim \cos(kx) \sim e^{ikx} = e^{ipx/\hbar}$. Consideration of the quantity $p\Psi \sim pe^{ipx/\hbar}$ leads to Eq. (7). But a harmonic wave extends to infinity and therefore gives a uniform probability distribution constant across all space. That does not make sense for describing a realistic particle, which can be at least approximately localized.

Borrowing a page from wave physics, quantum mechanics builds wave pulses by adding many harmonics to form a superposition of waves. Where the wave pulse is sharply peaked, the probability density for locating the particle there is large and the rest of the pulse damps out as one approaches infinity, as $P(-\infty,+\infty) = 1$ requires. The price paid for this formulation is that as a superposition of harmonics, there is no unique wavenumber for a wave pulse; therefore the approximately localized particle cannot have a unique momentum.

However, when the width of the wave pulse becomes very small, then the particle begins to approach the classical ideal of localization. Since probability implies an ensemble of measurements on identically prepared systems, the ensemble average of

dp/dt should equal the ensemble average of the Newtonian force. Ehrenfest showed this to be so by proving that

$$\langle \frac{-\partial U}{\partial x} \rangle = \langle \frac{dp}{dt} \rangle. \tag{8}$$

Yesterday's great discovery is today's homework exercise, and so it is with Ehrenfest's theorem. We do not *really know* Ehrenfest's theorem until we make it our own by proving it for ourselves. An efficient way to do that makes use of the following result for the ensemble average of the time rate of change of any quantity Q:

$$\langle \frac{dQ}{dt} \rangle = \langle \frac{\partial Q}{\partial t} \rangle + \frac{i}{\hbar} \langle (H, Q) \rangle \tag{9}$$

where $(H,Q) \equiv HQ - QH$, the "commutator" of H and Q. Before proving Ehrenfest's theorem, we must prove this identity; then Ehrenfest's theorem follows at once. To proceed, differentiate Eq. (4) with respect to time, which yields

$$\frac{d\langle Q \rangle}{dt} = \int_{-\infty}^{+\infty} \left\{ \frac{\partial \Psi^*}{\partial t} Q + \Psi^* \frac{\partial Q}{\partial t} \Psi + \Psi^* Q \frac{\partial \Psi}{\partial t} \right\} dx. \tag{10}$$

Use Eq. (7) to replace $\partial\Psi/\partial t$ with $-iH/\hbar$ (and likewise for its complex conjugate, noting that $H^* = H$ if $U^* = U$, which is necessary for $P(-\infty,+\infty) = 1$ to hold for all time), and obtain Eq. (9).

Ehrenfest's theorem follows by setting $Q = p$. Since momentum is proportional to the gradient, it is not explicitly a time-dependent operator; hence $\partial p/\partial t = 0$. Since p commutes with itself, $(H,p) = (U,p)$ so that

$$(Up - pU)\Psi = \frac{\hbar}{i}\left[U \frac{\partial \Psi}{\partial x} - \frac{\partial}{\partial x}(U\Psi)\right] - \frac{\hbar}{i}\frac{\partial U}{\partial x} \tag{11}$$

and Ehrenfest's theorem follows at once.

Ehrenfest's theorem is fundamental because it shows that quantum mechanics and Newtonian physics are consistent in the sense that the latter is contained within the more comprehensive former. Even though exact Newtonian trajectories do not exist in quantum mechanics (and only approximately so in the real world), the Newtonian paradigm still exists within quantum theory as an ensemble average.

PAUL EHRENFEST'S END

Our mention of Ehrenfest's contributions to physics must be limited to samples here. Our purpose was not to recite them all, but to show that Paul Ehrenfest earned the respect of other physicists professionally, in addition to having their respect as a human being and friend. An obituary for him published in the October 27, 1933, issue of *Science* lamented,[15]

> *The sudden news of the death of Professor Paul Ehrenfest, of the University of Leiden, has given his many friends all over the world a great shock of intense sorrow. It is difficult, with the pain lying as a stone on our hearts, to try to enumerate the special virtues of his lovable character and his great mind. Perhaps they were his honesty and his strong and humble desire to help where he could. Everybody could count on his help, and it was especially so for his students, not restricted to physics alone . . .*

How tragic, then, that in Ehrenfest's own mind, no matter what he accomplished, he always felt himself and his work to be of little worth, even though his colleagues held him in the highest esteem. His despondency became a downward spiral. To complicate his state of mind further, he and his wife Tatyana suffered a fateful partial estrangement,[16] and he had enormous trouble accepting the fact that one of his beloved children, his son Wassik, had Down syndrome and required lifelong clinical attention.

In May 1931 Ehrenfest told Bohr in a letter, "I have completely lost contact with theoretical physics. I cannot read anything anymore and feel myself incompetent to have even the most modest grasp about what makes sense in the flood of articles and books. Perhaps I cannot at all be helped anymore."[4]

In August of that same year he wrote a farewell letter to some of his former students:

> *. . . I have you much more to thank than you realize. Your affection, your consistent wish to give me confidence in myself made it possible until just recently for me to maintain my enthusiasm. Forgive me that it is now over.*

His last letter to some close friends, including Bohr and Einstein, evidently never sent, also carries the sad farewell message of one who has given up all hope:[4]

> *In recent years it has become ever more difficult for me to follow the developments in physics with understanding. After trying, ever more enervated and torn, I have finally given up in desperation. This made me completely weary of life . . .*

On September 25, 1933, Paul Ehrenfest saw no future for himself or his son Wassik. In the waiting room of a clinic where Wassik was being treated, Paul shot Wassik. Then he turned the gun on himself.

The following year Albert Einstein published a memoriam to his friend. He sadly wrote,[17]

> *He was not merely the best teacher in our profession whom I have ever known; he was also passionately preoccupied with the development and destiny of men, especially his students. . . . His students and colleagues in Leiden loved and esteemed him. They knew his utter devotion, his nature so wholly attuned to service and help. Should he not have been a happy man?*
>
> *In truth he felt unhappier than anyone else who was close to me. The reason was that he did not feel equal to the lofty task that confronted him. Of what use*

was it that everyone held him in esteem? His sense of inadequacy, objectively unjustified, plagued him incessantly, often robbing him of the peace of mind necessary for tranquil research. . . .

We whose lives have been enriched by the power and integrity of his spirit, the kindness and warmth of his rich mind, and not least his irrepressible humor and trenchant wit—we know how much his departure has impoverished us. He will live on in his students and in all whose aspirations were guided by his personality.

WHAT PAUL EHRENFEST CAN TEACH US BESIDES PHYSICS

Because he was a great teacher, I trust that Paul Ehrenfest would not mind us trying to extract insights from his story. Along with the statistical mechanics and quantum theory that we learn from him, he has even more important lessons to teach us about *life*. In a previous set of articles on depression and suicide among students,[18] we learned that students whose depression places them at risk for suicide often exhibit a pattern of behaviors. Paul Ehrenfest exhibited these behaviors. To him, ending it all made perfect *logical* sense because he saw no other way out. He was chronically melancholic, he had suffered loss at a young age, he talked about suicide before doing it. Above all, he was consistently harder on himself than was anyone else, underrating what he *did* have to offer, and exaggerating his own inadequacies. *If the little voice telling us we don't measure up is coming from inside our own heads, and not from our respected mentors who know us well, then that voice is a liar*. The tragedy for Paul and Wassik Ehrenfest, and their family and friends, is that Paul listened to that lying voice.

The study of physics is challenging, demanding, and frustrating at times. Sometimes we feel like quitting. Paul Ehrenfest's pain was very real, and that must be respected. He needed help and compassion, not judgment. I don't know how much help was available to Ehrenfest, or whether he availed himself of whatever help was available. But I do know that sources of help, with grace and understanding, *are* accessible today.

Two months ago I served as the faculty representative on the Student Development Committee of my university's board of trustees. This committee's responsibilities include the on-campus clinic and counseling center. The members learned that, last year, over five thousand appointments were made with the counseling center at a school of about 2,000 students (the center also serves the local community). Most of the appointments were about mental health issues. This generated focused discussion in the committee meeting. It was impressively clear that everyone took seriously their role in providing resources and an environment where struggling students could be met at the points of their needs. Each struggle is personal, but resources and caring people are willing and available. [For more on this, please see the story by Jim Bauer on page 18.]

The 2004 articles[18] cited studies showing that chronically depressed students typically imagine they are facing their difficulties alone. *You are not alone*. As a faculty member, I implore you: When you are stuck, when you see no way ahead, come talk to me. And I will seek you out if I detect something is going wrong. Together, we will work something out. Doing physics is not the most important task of the physics community. *Our most important task is to be a community*. [For thoughts about finding and building a community, see the story by Kendra Redmond on page 20.]

DOING PHYSICS IS NOT THE MOST IMPORTANT TASK OF THE PHYSICS COMMUNITY.

Our most important task is to be a community.

In most of my dealings with distressed students—and this was certainly true in my own experience as a distressed student—one's identity and self-worth as a person can too easily be equated to how one is doing in this major or in that course. In his farewell letter Ehrenfest said it had become "*ever more difficult for me to follow the developments in physics with understanding. . . . I have finally given up in desperation. This made me completely weary of life . . .*" Ehrenfest's personal struggles show that he was wrestling with more issues than dissatisfaction with his understanding of physics. But physics was an important part of who he was, and it's an important part of your life and mine, too.

In the 1971 movie *Le Mans*, Steve McQueen's character, Michael Delaney, when asked what's so important about racing as to risk one's life for it, replied that, to a race car driver, "Everything else is just waiting." Nobody consciously decides that his or her worth *as a person* depends on winning Le Mans or getting an A in quantum mechanics. We get drawn into performance-as-intrinsic-worth traps incrementally and unconsciously. Yes, we physics students, like race car drivers, should do everything we can to achieve excellence. We should take the *work* seriously, but we should not take *ourselves* seriously. My small grandchildren do not care what my grades were or how many papers I have published. But they do care—and they can tell—whether or not I am genuine. My worth as a *person* has nothing to do with physics. And as much as I love it, physics is only a small part of a very large world of experiences to appreciate.

Ehrenfest and Bohr met in 1918, in Leiden. M.J. Klein recalls how they "had much to talk about together—from the current problems of quantum theory to the Icelandic sagas . . . Their exchanges ranged over heaven and earth as Ehrenfest showed his new friend the treasures of the Dutch museums and the brilliant colors of the bulb fields."[4] Sometimes the museums and Icelandic sagas and the fields of flowers are more important than quantum theory.

As we pursue the elegant connections in physics, let us never let a stack of physics papers disconnect us from the great range of experiences that life has to offer:

Parchment—is that the sacred fount
From which you drink to still your thirst forever?
If your refreshment does not mount
From your own soul, you gain it never.
—Faust [19]

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I have, alas, studied philosophy,
Jurisprudence and medicine, too,
And, worst of all, theology
With keen endeavor, through and through—
And here I am for all my lore,
The wretched fool I was before . . .

Walter Kaufmann, tr., *Goethe's Faust* (Doubleday, Garden City, NY, 1963), p. 93.

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