Quantum Theory
lecture 6: time-dependent systems

- adiabatic theorem — B:12.7—KP:12.3.0,12.3.2
- Landau-Zener transitions — KP:12.3.1
- Berry phase — B:12.7—KP:12.3.3,12.3.4
- Dirac fermions in graphene

B = Ballantine — KP = Konishi & Paffuti

classical adiabatic theorem

$J = \oint p\,dq$ is an adiabatic invariant of the time-dependent Hamiltonian $H[p, q, \alpha(t)]$, meaning $J(T) - J(0) \sim \dot{\alpha}$ and not $\sim \alpha(T) - \alpha(0)$.

condition: frequency change in one period much smaller than frequency itself, so $|\dot{\omega}| \ll \omega^2$.

- harmonic oscillator (Lorentz pendulum)

$$J = 4\int_0^{\sqrt{2mE}/m\omega} \sqrt{2mE - m^2\omega^2x^2}\,dx = 2\pi E/\omega.$$

- 2D cyclotron motion in a varying field: enclosed flux constant

$$J = \oint (mv + qA)\cdot dq = mv\,2\pi l_c - q\Phi = 2qB\pi l_c^2 - q\Phi = q\Phi$$

- channel of varying width: collimation by a horn $\quad J = 2Wp\sin\theta$

quantum adiabatic theorem

quantization of adiabatic invariant (Ehrenfest): $J = \oint p\,dq = nh$

no transitions between energy levels if the Hamiltonian changes sufficiently slowly

condition: $\hbar\omega = E_{n+1} - E_n \equiv \delta \Rightarrow \hbar|\dot{\delta}| \ll \delta^2$

adiabaticity breaks down when levels cross, or almost cross (avoided level crossing, $\delta_{\text{min}} > 0$)

Landau-Zener transition: jump from one level to the other at the avoided crossing

probability $P_{\text{jump}} \ll 1$ if $\hbar|\dot{\delta}| \ll \delta^2_{\text{min}}$

$$P_{\text{jump}} = \exp\left(-\frac{\pi\delta^2_{\text{min}}}{2\hbar|\dot{\delta}|}\right)$$

exact for $H = \begin{pmatrix} \alpha t & \gamma \\ \gamma^* & \beta t \end{pmatrix}$

$\delta_{\text{min}} = 2|\gamma|, \dot{\delta} = \alpha - \beta$

$E_\pm = \frac{1}{2}(\alpha+\beta)t \pm \sqrt{|\gamma|^2 + \frac{1}{4}(\alpha-\beta)^2 t^2}$

Berry phase

adiabatic evolution with return to starting point ("cyclic"): final state can only differ from initial state by a phase factor (for a non-degenerate state)

$$\psi_{\text{final}} = e^{i\gamma_B} \exp\left(-\frac{i}{\hbar}\int_0^T E(t)\,dt\right)\psi_{\text{initial}}$$

dynamical phase factor, depending on the period T of the cycle

Berry phase γ_B independent on T ("geometric" phase)

$i\hbar\frac{d}{dt}|\psi(t)\rangle = H[\alpha(t)]\psi(t)$, eigenstate: $H[\alpha(t)]|\alpha(t)\rangle = E(t)|\alpha(t)\rangle$

substitute: $|\psi(t)\rangle = e^{i\gamma(t)}e^{-(i/\hbar)\int_0^t E(t')\,dt'}|\alpha(t)\rangle \Rightarrow \frac{d\gamma}{dt} = i\langle\alpha(t)|\frac{d}{dt}|\alpha(t)\rangle = i\frac{d\alpha}{dt}\langle\alpha(t)|\frac{\partial}{\partial\alpha}|\alpha(t)\rangle$

$$\gamma_B = i\oint\left\langle\alpha\left|\frac{\partial}{\partial\alpha}\right|\alpha\right\rangle d\alpha$$

example: Berry phase of a spin in a rotating magnetic field

$$H[\vec{B}(t)] = -\frac{\mu}{2}\vec{\sigma}\cdot\vec{B}(t) = -\frac{\mu}{2}\begin{pmatrix} B_z(t) & B_x(t) - iB_y(t) \\ B_x(t) + iB_y(t) & -B_z(t) \end{pmatrix}$$

$\vec{B}(t) = B_0(\sin\theta\cos\phi(t), \sin\theta\sin\phi(t), \cos\theta)$ rotates around the z-axis; consider the evolution of a spin-up eigenstate:

$$|\phi(t)\rangle = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi(t)}\sin(\theta/2) \end{pmatrix} \Rightarrow i\langle\phi|\frac{d}{d\phi}|\phi\rangle = -\sin^2(\theta/2)$$

$$\gamma_B = \pm 2\pi\sin^2(\theta/2)$$

Berry phase is one-half the solid angle covered by the rotating spin

Dirac fermions in graphene

see exercises 6.2 and 6.3