Quantum Theory
lecture 2: Symmetry

the force of symmetry

- conservation laws — B:3.8 — KP:5.2.0
- unitary & anti-unitary symmetries — B:3.1 — KP:5.2.3
- parity — B:13.1 — KP:5.2.2
- time-reversal & Kramers degeneracy — B:13.3 — KP:5.2.3
- Galilean invariance — B:3.2,3.3 — KP:5.2.4

B = Ballantine — KP = Konishi & Paffuti

conservation laws

Hamiltonian invariant under a unitary transformation

$$\hat{U}\hat{H}\hat{U}^\dagger = \hat{H} \Rightarrow [\hat{U}, \hat{H}] = 0$$

$$\hat{U} = e^{i\hat{A}} \Rightarrow i\hbar\frac{d\hat{A}}{dt} = [\hat{A}, \hat{H}] = 0$$

observable corresponding to $\hat{A}$ is conserved

(the Hermitian operator $\hat{A}$ is called the generator of the unitary symmetry $\hat{U}$)

translational symmetry: $\hat{U} = e^{i\hat{p}a/\hbar} \Rightarrow$ momentum conserved

see exercise 1.2, or consider an infinitesimal translation over a:

$\delta\hat{U} = 1 + \delta a\, \partial/\partial x$ with $\delta a = a/N$

then compose N such translations and take the limit $N \to \infty$

$$\hat{U} = \left(1 + \frac{a}{N}\frac{\partial}{\partial x}\right)^N = e^{a\partial/\partial x} = e^{ia\hat{p}/\hbar}$$

unitary & anti-unitary symmetries

Wigner's theorem: every symmetry S is either a unitary operator or an anti-unitary operator.

Symmetry: $|\langle S\psi|S\chi\rangle|^2 = |\langle\psi|\chi\rangle|^2$ for all states ψ, χ.
Unitary case: $S = U$, $UU^\dagger = U^\dagger U = 1$,
Anti-unitary case: $S = UK$, with K = complex conjugation
for example, time-reversal: $\psi^*(x, t) = \psi(x, -t)$

Sketch of a proof for a 2D Hilbert space (spin-$1/2$):
see ex. 1.4: $|\psi\rangle$ is associated with a vector $\vec{n}$ on the Bloch sphere

$$|\langle\psi|\chi\rangle|^2 = \tfrac{1}{2}(1 + \vec{n}_1 \cdot \vec{n}_2).$$

symmetry is angle-preserving mapping of the unit sphere on itself: only rotations ($S = U$) or rotations + reflection ($S = UK$).

parity (= spatial inversion)

see exercise 2.1

time-reversal & Kramers degeneracy

see exercise 2.2

Galilean invariance

see exercise 2.3