Quantum Theory
lecture 1: Basics

from wave mechanics to matrix mechanics

• position and momentum representation	B-4.1,5.1 — KP-7.1
• states and operators (bra-ket notation)	B-1.1,1.2 — KP-7.2
• unitary transformations	B-3.1 — KP-7.4
• Heisenberg equations of motion	B-3.7 — KP-7.5
• uncertainty relation	B-8.4 — KP-7.6
• pure states and mixtures, density matrix	B-2.3 — KP-7.7

B = Ballantine (mainly formalism)

KP = Konishi & Paffuti (mainly applications)

position & momentum representation

position operator: $\hat{q}\psi(q) = q\psi(q)$

eigenstate $\delta(q - q_0)$ at eigenvalue q_0
$\hat{q}\delta(q - q_0) = q\delta(q - q_0) = q_0\delta(q - q_0)$

momentum operator: $\hat{p}\psi(q) = -i\hbar\frac{\partial}{\partial q}\psi(q)$

eigenstate $\psi_p(q) = \frac{1}{\sqrt{2\pi\hbar}}e^{ipq/\hbar}$ because $\hat{p}\psi_p(q) = p\psi_p(q)$
normalization: $\int dq\, \psi^*_{p'}(q)\psi_p(q) = \delta(p - p')$

recall: $\int_{-\infty}^{\infty} dx\, e^{ikx} = 2\pi\delta(k)$

$$\phi(p) = \int dq\, \psi^*_p(q)\psi(q) = \frac{1}{\sqrt{2\pi\hbar}}\int dq\, e^{-ipq/\hbar}\psi(q)$$

momentum representation: $\hat{p}\phi(p) = p\phi(p)$, $\hat{q}\phi(p) = i\hbar\frac{\partial}{\partial p}\phi(p)$

states & operators (bra-ket notation)

state: $|\psi\rangle$ (ket = column vector) and $\langle\psi|$ (bra = row vector)
scalar product: $\langle\chi|\phi\rangle = \langle\phi|\chi\rangle^*$
orthonormal set: $\langle\phi_n|\phi_m\rangle = \delta_{nm}$

$$\text{completeness: } |\psi\rangle = \sum_n |\phi_n\rangle\langle\phi_n|\psi\rangle \Rightarrow \sum_n |\phi_n\rangle\langle\phi_n| = \hat{1}$$

"resolution of the identity"

operator: $\langle\chi|A\phi\rangle = \langle\chi|A|\phi\rangle$, $\langle A\chi|\phi\rangle = \langle\chi|A^\dagger|\phi\rangle$
Hermitian conjugate or adjoint operator: $(A^\dagger)_{nm} = A^*_{mn}$
self-adjoint (Hermitian): $A^\dagger = A$ (real eigenvalues, observable)

from q to p representation: $\psi(q) = \langle q|\psi\rangle$, $\phi(p) = \langle p|\psi\rangle$

$$\langle p|\psi\rangle = \int dq\, \langle p|q\rangle\langle q|\psi\rangle \Rightarrow \langle p|q\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-ipq/\hbar}$$

unitary transformations

$$\langle \hat{U}\phi|\hat{U}\chi\rangle = \langle\phi|\chi\rangle \Rightarrow \hat{U}\hat{U}^\dagger = \hat{U}^\dagger\hat{U} = \hat{1}$$

or $U^{-1} = U^\dagger$ unitary operator

example: $\hat{U} = e^{i\hat{A}}$ with $\hat{A}$ Hermitian

eigenvalues on the unit circle in the complex plane

$\phi' = \hat{U}\phi$, $\chi' = \hat{U}\chi$ is a change of basis for the states, what is the corresponding basis change for the operators?

$$\langle\phi|\hat{O}|\chi\rangle = \langle\phi'|\hat{U}\hat{O}\hat{U}^\dagger|\chi'\rangle \Rightarrow \hat{O}' = \hat{U}\hat{O}\hat{U}^\dagger$$

check that commutator $[\hat{q},\hat{p}] = i\hbar$ is unchanged upon unitary transformation

Heisenberg equation

solution of Schrödinger equation is a unitary transformation

$$i\hbar\frac{\partial}{\partial t}\psi(t) = \hat{H}\psi(t) \Rightarrow \psi(t) = e^{-i\hat{H}t/\hbar}\psi(0)$$

Schrödinger picture: time dependence in states

Heisenberg picture: time dependence in operators

$$\langle\phi(t)|\hat{O}|\chi(t)\rangle = \langle\phi|\hat{O}(t)|\chi\rangle$$

with $|\psi\rangle \equiv |\psi(0)\rangle$ and $\hat{O}(t) = e^{i\hat{H}t/\hbar}\hat{O}e^{-i\hat{H}t/\hbar}$

$$i\hbar\frac{d}{dt}\hat{O} = \hat{O}\hat{H} - \hat{H}\hat{O} = [\hat{O}, \hat{H}]$$

Heisenberg equation of motion

uncertainty relation

see exercise 1.6

pure & mixed states, density matrix

see exercise 1.3