6 Time-dependent quantum systems

6.1 Adiabatic and sudden approximations

A particle is in the ground state of the potential well confined by infinite barriers at $x = 0$ and $x = L$.

a) Suppose that the barrier at $x = L$ is moved *slowly* to $x = 2L$. Calculate in the adiabatic approximation the energy and wave function of the particle in the wider potential well. How much work is done in this process?

b) How slow should the barrier be moved to justify the adiabatic approximation?

c) Now suppose that the barrier at $x = L$ is moved *suddenly* at $t = 0$ to $x = 2L$. In the "sudden approximation" we assume that the wave function of the particle remains unchanged. What is then the probability that the particle remains in the ground state?

d) Why are we not allowed to make the sudden approximation when the barrier at $x = L$ is moved suddenly to $x = L/2$?

6.2 Berry phase in graphene

Electrons in graphene (a carbon monolayer) are described by a two-component wave function $\psi = (\psi_A, \psi_B)$. The labels A, B refer to the two sublattices of the honeycomb lattice of carbon atoms. The Hamiltonian is a 2×2 matrix,

$$H = v \begin{pmatrix} 0 & p_x - i p_y \\ p_x + i p_y & 0 \end{pmatrix}.$$

The velocity $v = 10^6$ m/s is independent of energy.

a) Derive the relation $E^2 = v^2 p^2$ between energy and momentum (the socalled dispersion relation). *Hint:* Try squaring H.
Fermions with this dispersion relation are known as Dirac fermions. Do you known of other particles that obey the same relation?

b) Verify the following expression for the wave function

$$\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm e^{i\phi} \end{pmatrix} e^{i\boldsymbol{p}\cdot\boldsymbol{r}/\hbar},$$

where ϕ is the angle of the momentum $\boldsymbol{p}$ with the x-axis. When should you take the + sign and when the − sign?

c) Assume that the electron moves adiabatically along a closed trajectory C in the x-y plane. Calculate the Berry phase from its definition

$$\gamma_{\mathrm{B}} = i \oint_C dt \left\langle \psi(t) \middle| \frac{\partial}{\partial t} \psi(t) \right\rangle .$$

The result $\gamma_{\mathrm{B}} = \pi$ is the same as that of a spin-1/2 rotating over 360°. The sublattice degree of freedom A, B is therefore sometimes called a "pseudospin".

One way to make the electron execute such a closed trajectory is to apply a perpendicular magnetic field $\boldsymbol{B} = B\hat{z}$. We then need to distinguish the mechanical momentum $\boldsymbol{p}_{\mathrm{mech}}$ from the canonical momentum $\boldsymbol{p} = \boldsymbol{p}_{\mathrm{mech}} + e\boldsymbol{A}$. The cyclotron radius $l_{\mathrm{c}} = p_{\mathrm{mech}}/eB$ and the energy $E = \pm v p_{\mathrm{mech}}$ are given by the *mechanical* momentum.

$*$*)* As an extra question, show that Dirac fermions in graphene follow the same cyclotron orbit in a magnetic field as ordinary electrons do, using the classical equation of motion in a force field $\boldsymbol{F}$,

$$\frac{d^2}{dt^2} r_i = \sum_j \frac{\partial^2 E}{\partial P_i \partial P_j} F_j = \frac{v}{P} \sum_j \left(\delta_{ij} - \frac{P_i P_j}{P^2} \right) F_j .$$

(We abbreviated $\boldsymbol{p}_{\mathrm{mech}} = \boldsymbol{P}$ and $P = |\boldsymbol{P}|$.) Note that the Lorentz force is perpendicular to $\boldsymbol{P}$. In the next exercise we'll see that the motion in an *electric* field is completely different.

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d) Use the quantization of enclosed flux from exercise 5.2.c to argue that the quantization rule in graphene is

$$\Phi_n = n\frac{h}{e}, \quad n = 0, 1, 2, \ldots$$

Explain the absence of the 1/2 offset.

e) Calculate that the Landau levels in graphene are given by

$$E_n^{\pm} = \pm\sqrt{2n}\,\frac{\hbar v}{l_{\mathrm{m}}}, \quad n = 0, 1, 2, \ldots$$

in terms of the socalled magnetic length $l_{\mathrm{m}} = \sqrt{\hbar/eB}$. Once again, this semiclassical result is actually exact.

The difference with the usual sequence $(n + 1/2)\hbar\omega_{\mathrm{c}}$ was first observed in 2005 by Geim, Novoselov, and Kim (and two of them won the Nobel prize a few years later).

6.3 Klein tunneling

Because the velocity of Dirac fermions is energy independent, electrons in graphene can be deflected by an electric field but they cannot be stopped. Let us investigate this for a uniform field $\boldsymbol{F} = F\hat{x}$, corresponding to the linear potential $V(x) = -Fx$. The Hamiltonian is

$$H = \begin{pmatrix} -Fx & v(p_x - ip_y) \\ v(p_x + ip_y) & -Fx \end{pmatrix}.$$

An electron is incident from $x = -\infty$ on the potential barrier and we wish to know the probability T that it is transmitted to $x = \infty$. We assume that the electron has energy $E = 0$.

a) Why can we assume this without any loss of generality?

b) Show that the equation $H\psi = 0$ can be written in momentum representation as

$$i\hbar\frac{\partial\psi}{\partial p_x} = \frac{v}{F}(p_x\sigma_x + p_y\sigma_y)\psi.$$

If we interpret $p_x \equiv t$ as "time", this looks like a Schrödinger equation for a spin-1/2 particle moving in the "time"-dependent potential

$$U(t) = \frac{v}{F}\begin{pmatrix} 0 & t - ip_y \\ t + ip_y & 0 \end{pmatrix}.$$

c) Plot the two eigenvalues $E_\pm(t)$ of $U(t)$ as a function of t, for $p_y = 0$ and for some nonzero value of p_y.
Explain why the transmission of the particle through the barrier corresponds to a Landau-Zener transition from E_+ to E_- with increasing t.

d) Use the Landau-Zener formula to obtain the transmission probability

$$T = \exp\left(-\frac{\pi v p_y^2}{\hbar F}\right).$$

Klein tunneling refers to tunneling with unit transmission probability for normal incidence ($p_y = 0$), no matter how high the barrier.