3 Fermions and bosons

exercises 3.1 and 3.4 by Dr. Peter Denteneer and Dmitry Pikulin

3.1 Second quantization

Consider a system of spinless bosons in a (large) volume V. We will investigate how the difference between an *external* (one-body) potential and an *interaction* (two-body) potential appears when the Hamiltonian is written in terms of creation and annihilation operators (second quantization). Recall the definition of field and number operators,

$$\hat{\psi}(\vec{r}) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}\, \hat{a}_{\vec{k}}, \;\; \hat{n}(\vec{r}) = \hat{\psi}^{\dagger}(\vec{r})\, \hat{\psi}(\vec{r}).$$

(The integral over continuous $\vec{k} = \vec{p}/\hbar$ in infinite space, with a delta-function normalization, is replaced here by a sum over discrete $\vec{k}$ in a finite volume V.)

a) In the case of an external potential $u(\vec{r})$, show that the operator for the total potential energy,

$$\hat{U} = \int_V d\vec{r}\;\hat{n}(\vec{r})\, u(\vec{r}) \;,$$

can be expressed as follows in terms of creation and annihilation operators:

$$\hat{U} = \sum_{\vec{q}} \frac{u_{\vec{q}}}{\sqrt{V}} \sum_{\vec{m}} \hat{a}^{\dagger}_{\vec{m}+\vec{q}}\, \hat{a}_{\vec{m}} \;,$$

where

$$u_{\vec{q}} \equiv \frac{1}{\sqrt{V}} \int_V d\vec{r}\; e^{-i\vec{q}\cdot\vec{r}}\; u(\vec{r})$$

are the Fourier components of the potential $u(\vec{r})$.

Let $v(r_{12})$ be a short-range interaction potential, dependent only on the separation $r_{12} = |\vec{r}_1 - \vec{r}_2|$.

b) Show that:

$$\begin{aligned} v(\vec{k}\vec{\ell},\vec{m}\vec{n}) &= \frac{1}{V^2} \int d\vec{r}_1 d\vec{r}_2\, e^{-i(\vec{k}\cdot\vec{r}_1 + \vec{\ell}\cdot\vec{r}_2 - \vec{m}\cdot\vec{r}_1 - \vec{n}\cdot\vec{r}_2)}\; v(r_{12}) \\ &= \delta_{\vec{k}+\vec{\ell},\vec{m}+\vec{n}}\, \frac{1}{\sqrt{V}}\, v_{\vec{k}-\vec{m}}, \end{aligned}$$

where $v_{\vec{q}}$ are the Fourier components of $v(r)$.

c) Show that the operator for the total interaction energy,

$$\hat{H}^{(\mathrm{int})} = \frac{1}{2}\sum_{i\neq j} v\left(|\hat{\vec{r}}^{(i)} - \hat{\vec{r}}^{(j)}|\right),$$

is given in terms of creation and annihilation operators by

$$\hat{H}^{(\mathrm{int})} = \sum_{\vec{q}} \frac{v_{\vec{q}}}{2\sqrt{V}} \sum_{\vec{m},\vec{n}} \hat{a}^{\dagger}_{\vec{m}+\vec{q}}\,\hat{a}_{\vec{m}}\,\hat{a}^{\dagger}_{\vec{n}-\vec{q}}\,\hat{a}_{\vec{n}}.$$

3.2 Coherent states

Coherent states are eigenstates of the bosonic annihilation operator. They are called that way, because a laser (a source of coherent radiation) produces a coherent state of photons at a particular frequency.

a) Show that the state

$$|\beta\rangle = e^{-|\beta|^2/2} e^{\beta a^{\dagger}}|0\rangle$$

is a coherent state, $a|\beta\rangle = \beta|\beta\rangle$. Verify that this state is normalized to unity.

b) There is a very simple formula for expectation values of operators of the form $A = f(a^{\dagger})g(a)$, where all creation operators are to the left of the annihilation operators. (This is called the "normal order"):

$$\langle\beta|f(a^{\dagger})g(a)|\beta\rangle = f(\beta^*)g(\beta).$$

Why is this true?

c) Prove that the probability $P(n)$ that the coherent state contains n photons is a Poisson distribution,

$$P(n) = e^{-|\beta|^2}\frac{|\beta|^{2n}}{n!}.$$

Because this is the distribution of classical independent particles, a coherent state is also referred to as a "classical" state of the electromagnetic field.

d) Another way to see the (almost) classical nature of a coherent state, is to calculate the uncertainty of the canonically conjugate operators

$$x = 2^{-1/2}(a + a^{\dagger}), \;\; p = 2^{-1/2} i(a^{\dagger} - a).$$

Show that the product of $\langle(\Delta x)^2\rangle$ and $\langle(\Delta p)^2\rangle$ takes the minimal value that is consistent with the Heisenberg uncertainty principle.

3.3 Bogoliubov transformations

The Bogoliubov transformation is a linear transformation of the creation and annihilation operators that preserves their commutation relation. We examine this for a pair of fermionic creation and annihilation operators $\hat{a}_\alpha^\dagger$ and $\hat{a}_\alpha$, with $\alpha = \pm$ labeling spin and/or momentum. The commutation relations are

$$\hat{a}_\alpha \hat{a}_{\alpha'} + \hat{a}_{\alpha'} \hat{a}_\alpha = 0 \text{ and } \hat{a}_\alpha \hat{a}^\dagger_{\alpha'} + \hat{a}^\dagger_{\alpha'} \hat{a}_\alpha = \delta_{\alpha\alpha'} \hat{1}.$$

Consider the linear transformation

$$\hat{b}_\alpha = u_\alpha \hat{a}_\alpha - v_\alpha \hat{a}^\dagger_{-\alpha},$$

with real coefficients u_α and v_α.

a) What is the corresponding expression for $\hat{b}_\alpha^\dagger$?

b) Show that the fermionic commutation relations are preserved if u_α and v_α satisfy

$$u_\alpha^2 + v_\alpha^2 = 1, \;\; u_\alpha v_{-\alpha} + u_{-\alpha} v_\alpha = 0.$$

The particle created by the operator $\hat{b}_\alpha^\dagger$ is a fermion but it is not an electron. It is called a Bogoliubov quasiparticle (or “Bogoliubon”).

We may choose $u_\alpha = u_{-\alpha} \equiv u$ and $v_\alpha = -v_{-\alpha} \equiv v$, so that the Bogoliubov transformation takes the form

$$\hat{b}_+ = u\hat{a}_+ - v\hat{a}_-^\dagger, \;\; \hat{b}_- = u\hat{a}_- + v\hat{a}_+^\dagger.$$

c) Derive the inverse transformation, expressing $\hat{a}_\alpha$ in terms of $\hat{b}_\alpha$ and $\hat{b}^\dagger_{-\alpha}$.

Bogoliubov and Valatin used this transformation to calculate the excitation spectrum of a superconductor. The Hamiltonian is

$$\hat{H} = \xi(\hat{a}_+^\dagger \hat{a}_+ + \hat{a}_-^\dagger \hat{a}_-) + \Delta(\hat{a}_+^\dagger \hat{a}_-^\dagger + \hat{a}_- \hat{a}_+),$$

with real coefficients ξ and Δ. This Hamiltonian represents the mean-field BCS theory of superconductivity, after Bardeen, Cooper, and Schrieffer. The single-electron energy ξ measures the deviation from the Fermi level, $\xi = p^2/2m - E_{\rm F}$.

d) Notice that $\hat{H}$ does not conserve the number of particles, but it does conserve the parity of the particle number. Which terms can change the number of particles by ± 2 ? A pair of fermions is called a Cooper pair and Δ is called the pair potential.

e) Write $\hat{H}$ in terms of the Bogoliubov operators $\hat{b}_\alpha$ and observe what happens if you take the special choice

$$v^2 = 1 - u^2 = \frac{1}{2} - \frac{\xi}{2\sqrt{\xi^2 + \Delta^2}}.$$

Explain that the energy $E = \sqrt{\xi^2 + \Delta^2}$ is the excitation energy of a Bogoliubov quasiparticle. Plot it as a function of ξ and discuss the result.

3.4 Majorana fermions

Following Kitaev, consider spin-polarized fermions on a chain of N sites with Hamiltonian

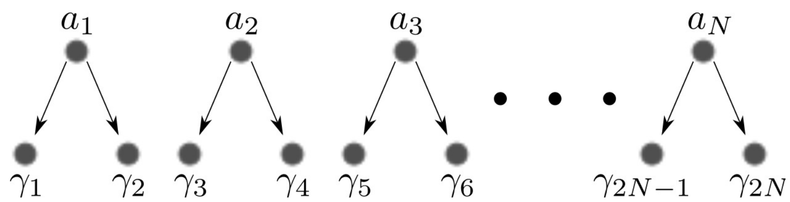
$$\hat{H} = \sum_{j=1}^{N-1} \left[t(\hat{a}_j^\dagger \hat{a}_{j+1} + \hat{a}_{j+1}^\dagger \hat{a}_j) + \Delta(\hat{a}_j \hat{a}_{j+1} + \hat{a}_{j+1}^\dagger \hat{a}_j^\dagger) \right] - \mu \sum_{j=1}^{N} \hat{a}_j^\dagger \hat{a}_j,$$

where t is the hopping amplitude between neighbouring sites, μ is the chemical potential, and Δ is the pair potential.

We make the transformation

$$\gamma_{2j-1} = a_j + a_j^\dagger \text{ and } \gamma_{2j} = -i(a_j - a_j^\dagger).$$

indicated in the figure. The γ operators are called "Majorana operators" and the quasiparticles they represent are called "Majorana fermions".



a) Compute $\gamma_n^\dagger$ and compare it to γ_n. Explain why it is said that a Majorana fermion "is its own antiparticle".

b) What are the commutation relations of the operators γ_j ? Evaluate γ_j^2.

c) Rewrite the Hamiltonian in terms of the Majorana operators.

d) Consider the special case $\Delta = -t > 0$, $\mu = 0$. Are there Majorana operators that are absent from the Hamiltonian? Can you make a fermionic operator out of the absent ones? Where does this fermion reside on the chain?

In this case the Hamiltonian has two degenerate states, distinguished by the occupation number of the fermionic state. This degeneracy has been proposed by A. Yu. Kitaev as a way to store information in a quantum computer. Because the information is distributed over the two ends of the chain, it is believed to be less sensitive to external perturbations than information that is stored locally.