2 Symmetries

2.1 Parity

The parity operator $\mathscr{P}$ is a unitary operator that changes the sign of the position and momentum operators:

$$\mathscr{P}\hat{q}\mathscr{P}^{-1} = -\hat{q} \text{ and } \mathscr{P}\hat{p}\mathscr{P}^{-1} = -\hat{p}.$$

a) Starting from the commutator $[\hat{q}, \hat{p}] = i\hbar$, show that $\mathscr{P}$ cannot be anti-unitary.

b) Explain why we may assume, without loss of generality, that $\mathscr{P}^2 = 1$. What does this imply for the eigenvalues of $\mathscr{P}$?
(The two possibilities are referred to as eigenstates of *odd* or *even* parity.)

c) The electrical dipole moment of a set of N particles, with charge e_n at positions $\boldsymbol{q}_n$, is given by $\hat{\boldsymbol{d}} = \sum_{n=1}^{N} e_n \hat{\boldsymbol{q}}_n$. If the system has inversion symmetry, the Hamiltonian $\hat{H}$ will commute with $\mathscr{P}$. Now show that the expectation value of the dipole moment must vanish in a *nondegenerate* eigenstate $|\Psi\rangle$ of $\hat{H}$.
Hint: Derive first that $\mathscr{P}|\Psi\rangle = \pm|\Psi\rangle$.

2.2 Time-reversal symmetry

The time-reversal operator $\mathscr{T}$ changes the sign of the momentum operator but leaves the position operator unaffected:

$$\mathscr{T}\hat{q}\mathscr{T}^{-1} = \hat{q} \text{ and } \mathscr{T}\hat{p}\mathscr{T}^{-1} = -\hat{p}.$$

a) Starting from the commutator $[\hat{q}, \hat{p}] = i\hbar$, show that $\mathscr{T}$ cannot be unitary.

b) We can identify $\mathscr{T}$ with the operator K of complex conjugation in position representation, $K\psi(x) = \psi^*(x)$. More generally, we can include a unitary operator $\hat{U}$ that commutes with $\hat{q}$ and $\hat{p}$, to arrive at the anti-unitary time-reversal operator

$$\mathscr{T} = \hat{U}K.$$

Explain why $\mathscr{T}^2$ must equal either $+1$ or -1.

c) Time reversal for a spin-1/2 particle (electron) should also invert the spin: $\mathscr{T}\sigma_n\mathscr{T}^{-1} = -\sigma_n$ for each Pauli matrix $\sigma_1, \sigma_2, \sigma_3$. Show that this works if we take $U = \sigma_2$. What is $\mathscr{T}^2$ in this case?

d) Show that $\mathcal{T}$ commutes with the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m}\sigma_0 + V(\hat{q})\sigma_0 + \hat{p}(a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3).$$

(The vector $\vec{a} = (a_1, a_2, a_3)$ describes spin-orbit coupling.)

e) A consequence of time-reversal symmetry is that the energy eigenvalues are all twofold degenerate (Kramers degeneracy). Prove this in two steps: First show that if $|\psi\rangle$ is an eigenstate of $\hat{H}$ at eigenvalue E, then also $\mathcal{T}|\psi\rangle$ is an eigenstate with the same eigenvalue. Then show that these two eigenstates are linearly independent, meaning that $|\psi\rangle = \lambda\mathcal{T}|\psi\rangle$ leads to a contradiction.

2.3 Galilean invariance

In classical mechanics, the transformation to a new coordinate system that moves with velocity v is called a Galilean transformation. Position and momentum are transformed as $x \mapsto x - vt$, $p \mapsto p - mv$.

a) In quantum mechanics, operators are transformed by a unitary transformation $\hat{O} \mapsto \hat{U}\hat{O}\hat{U}^{-1}$. Show that the unitary operator $\hat{U}$ that performs the Galilean transformation on the position and momentum operators $\hat{x}$ and $\hat{p}$ is

$$\hat{U} = e^{i\hat{G}},\ \ \hat{G} = \frac{1}{\hbar}v(m\hat{x} - \hat{p}t).$$

b) Evaluate the Heisenberg equation of motion of $\hat{G}$ for the Hamiltonian $\hat{H} = \hat{p}^2/2m$ of a free particle, to show that $d\hat{G}/dt = 0$.